From groups to semigroups and groupoids

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SUMMARY

(1) Semigroups

- Cancellative semigroups - Examples
- Inverse semigroups - Examples
- Cancellative semigroups $\longrightarrow$ inverse semigroups

(2) C^*-algebras of semigroups

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- Cancellative semigroups - Examples
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(2) C^*-algebras of semigroups

(3) Étale groupoids

- Examples
- Inverse semigroups $\longrightarrow$ étale groupoids

(4) Weak containment and amenability

(5) Exactness

CANCELLATIVE SEMIGROUPS

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Examples :

- $P = \mathbb{N} \subset \mathbb{Z}$ additive
- $P = \mathbb{N}^{x} \subset \mathbb{Q}^{x}$ multiplicative
- $\mathbb{P}_n = \overbrace{\mathbb{N} * \mathbb{N} * \cdots * \mathbb{N}}^{n \text{ times}} \subset \mathbb{F}_n$
- $P = \mathbb{N} \rtimes \mathbb{N}^{x} \subset \mathbb{Q} \rtimes \mathbb{Q}^{x}$
- $P = R \rtimes R^{x} \subset \mathcal{Q}(R) \rtimes \mathcal{Q}(R)^{x}$ where R is an integral domain and $\mathcal{Q}(R)$ its field of fractions
- ......

INVERSE SEMIGROUPS

An **inverse semigroup** S is a semigroup such that for each $s \in S$ there exists a unique $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$.

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$\rightsquigarrow$ The set $E = \{s \in S : s^2 = s\}$ of **idempotents** is a commutative sub-semigroup of S.

Examples :

- Discrete groups = inverse semigroups with a unique idempotent
- Cuntz and Cuntz-Krieger inverse semigroups
- Graph inverse semigroups
- Tiling inverse semigroups
- Free inverse semigroups
- Inverse semigroups of partial isometries in a Hilbert space

Examples :

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- Inverse hull of a cancellative semigroup.

Let P be a left cancellative semigroup. For $p \in P$, we denote by $L_p \in \mathrm{Inv}(P)$ the bijection $x \mapsto px$ from P onto pP.

The **left inverse hull** of P is the inverse sub-semigroup $S(P)$ of $\mathrm{Inv}(P)$ generated by the partial bijections L_p, $p \in P$.

C^*-ALGEBRAS OF AN INVERSE SEMIGROUP S

$\ell^1(S)$ is a Banach $*$-algebra with respect to the operations

$$(f * g)(s) = \sum_{uv=s} f(u)g(v), \quad f^*(s) = \overline{f(s^*)}$$

The **full C^*-algebra** $C^*(S)$ of S is the enveloping C^*-algebra of $\ell^1(S)$. It is the universal C^*-algebra for the representations of S by partial isometries.

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The **left regular representation** $\pi_2 : S \to \mathcal{B}(\ell^2(S))$ is defined by

$$\pi_2(s)\delta_t = \delta_{st} \text{ if } (s^*s)t = t, \quad \pi_2(s)\delta_t = 0 \text{ otherwise}$$

The **reduced C^*-algebra** $C_r^*(S)$ of S is the C^*-algebra generated by $\pi_2(S)$.

π_2 is faithful on $\ell^1(S)$, and so S is isomorphic to an inverse semigroup of partial isometries.

C^*-ALGEBRAS OF A SEMIGROUP P embeddable in a group G

The **reduced** or **Toeplitz** C^***-algebra** $C_r^*(P)$ is the sub-C^*-algebra of $\mathcal{B}(\ell^2(P))$ generated by the isometries $V_s : \delta_t \mapsto \delta_{st}$, $s \in P$.

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The **Wiener-Hopf** C^***-algebra** $W(P, G)$ is the sub-C^*-algebra of $\mathcal{B}(\ell^2(P))$ generated by the operators $W_g = E_P \lambda_g E_P$, $g \in G$, where λ is the left regular representation of G, and $E_P : \ell^2(G) \to \ell^2(P)$ is the orthogonal projection.

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We have

$$C^*(P) = C^*(S(P)) \twoheadrightarrow C_r^*(S(P)) \stackrel{h}{\twoheadrightarrow} C_r^*(P) \stackrel{\kappa}{\hookrightarrow} W(P, G)$$

where $h(\pi_2(L_p)) = V_p$ for $p \in P$.

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When is h injective ?, κ surjective ?

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What are the relations between :

(1) P left amenable

(2) $C_r^*(P)$ nuclear

(3) $C^*(P) = C_r^*(P)$ (**weak containment property**)

$$C^*(P) = C^*(S(P)) \twoheadrightarrow C^*_r(S(P)) \overset{h}{\twoheadrightarrow} C^*_r(P) \overset{\kappa}{\hookrightarrow} W(P, G)$$

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For $P = \mathbb{P}_n$, one has the exact sequence

$$0 \to \mathcal{K}(\ell^2(\mathbb{P}_n)) \to C^*_r(\mathbb{P}_n) \to \mathcal{O}(n) \to 0$$

Therefore $C^*_r(\mathbb{P}_n)$ is nuclear. Moreover $\mathbb{P}_n$ has the weak containment property.

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(1) $\Rightarrow$ (2) but (2) $\not\Rightarrow$ (1) ($\mathbb{P}_n$ is not left amenable).

If h is injective, we have (2) $\Rightarrow$ (3), but (3) $\Rightarrow$ (2) is open.

ETALE GROUPOIDS

A **groupoid** $\mathcal{G}$ is a small category in which every morphism is invertible.

We have a set $\mathcal{G}^{(0)} \subset \mathcal{G}$ of objects (or units), and four maps

$$r : \mathcal{G} \to \mathcal{G}^{(0)}, \quad s : \mathcal{G} \to \mathcal{G}^{(0)}, \quad m : \mathcal{G}^{(2)} \to \mathcal{G}, \quad i : \mathcal{G} \to \mathcal{G},$$

called source, target, multiplication and inverse where

$$\mathcal{G}^{(2)} = \{(\gamma, \gamma') : s(\gamma) = r(\gamma')\}$$

(composable morphisms). We write $m(\gamma, \gamma') = \gamma\gamma'$ and $i(\gamma) = \gamma^{-1}$.

These structure maps are required to obey obvious axioms.

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These structure maps are required to obey obvious axioms.

A **locally compact groupoid** is a groupoid endowed with a locally compact topology such that the structure maps are continuous. It is said to be **étale** if s and r are local homeomorphisms. Then $\mathcal{G}^{(0)}$ is a closed and open subset of $\mathcal{G}$ and the fibers $\mathcal{G}^x = r^{-1}(x)$, $\mathcal{G}_x = s^{-1}(x)$ are discrete for $x \in \mathcal{G}^{(0)}$.

Examples of étale groupoids :

- Locally compact spaces $X = \mathcal{G} = \mathcal{G}^{(0)}$.
- Discrete groups.
- **Bundle of discrete groups.** They are étale groupoids such that $r = s$. Then $r^{-1}(x)$ is a group for each unit x.

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- **Bundle of discrete groups.** They are étale groupoids such that $r = s$. Then $r^{-1}(x)$ is a group for each unit x.
- **Groupoids of partial actions.** A partial action of a discrete group G on a locally compact space X is a family $(\theta_g)_{g \in G}$ of homeomorphisms $\theta_g : D_g \to D_{g^{-1}}$ between open subsets of X such that $\theta_e = \mathrm{Id}_X$, θ_{gh} extends $\theta_g \circ \theta_h$. Then

$$X \rtimes G = \{(x, g, y) \in X \times G \times X : g \in G, y \in D_g, x = gy\}$$

is an étale groupoid, with the induced topology, and $r(x, g, y) = (x, e, x)$, $s(x, g, y) = (y, e, y)$, $(x, g, y)(y, h, z) = (x, gh, z)$, $(x, g, y)^{-1} = (y, g^{-1}, x)$.

Examples of étale groupoids :

- **Groupoids of inverse semigroup actions.** An action of an inverse semigroup S on a locally compact space X is an homomorphism θ from S into the inverse semigroup $\mathrm{Inv}(X)$ such that for every $a \in S$, the domain D_a of θ_a is open and θ_a is an homeomorphism from D_a onto D_{a^*}.

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- **Groupoids of inverse semigroup actions.** An action of an inverse semigroup S on a locally compact space X is an homomorphism θ from S into the inverse semigroup $\mathrm{Inv}(X)$ such that for every $a \in S$, the domain D_a of θ_a is open and θ_a is an homeomorphism from D_a onto D_{a^*}.

The groupoid $\mathcal{G}$ of germs of θ is the quotient of $\{(a, x) : a \in S, x \in D_a\}$ with respect to the equivalence relation

$$(a, x) \sim (b, y) \Leftrightarrow x = y \text{ and } \exists e \in E \text{ with } x \in D_e,\ ae = be.$$

We have

$$s([a, x]) = [e, x] \equiv x,\ r([a, x]) = \theta_a(x),\ [a, x][b, y] = [ab, y], \ldots.$$

(where e is any idempotent such that $x \in D_e$). A basis of the (not always Hausdorff) topology of $\mathcal{G}$ is given by the

$$\Theta(a, U) = \{[a, x], x \in U\}$$

for $a \in S$ and U open subset of D_a.

GROUPOID $\mathcal{G}_S$ OF AN INVERSE SEMIGROUP S.

Let E be the sub-semigroup of idempotents in S, and let $\widehat{E} \subset \{0,1\}^E$ be the locally compact totally discontinuous set of nonzero elements χ satisfying $\chi(ef) = \chi(e)\chi(f)$ for all $e, f \in E$.

S acts on $\widehat{E}$ as follows. For $a \in S$,

$$D_a = \{\chi : \chi(a^*a) = 1\}, \ \theta_a(\chi)(e) = \chi(a^*ea).$$

$\mathcal{G}_S$ is the groupoid associated to this action.

A result of Khoshkam-Skandalis shows that for many inverse semigroups, $\mathcal{G}_S$ is Hausdorff and Morita equivalent to a group action.

C^*-ALGEBRAS OF A GROUPOID[1] $\mathcal{G}$

• **Groupoid $*$-algebra** $C_c(\mathcal{G})$ **:** it is the $*$-algebra of continuous compactly supported functions on $\mathcal{G}$ with product and $*$-operation given by

$$(f * g)(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2), \quad f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

• **Full C^*-algebra** $C^*(\mathcal{G})$ **:** it is the universal completion of $C_c(\mathcal{G})$ with respect to its representations.

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• **Reduced C^*-algebra $C_r^*(\mathcal{G})$.** For $x \in X = \mathcal{G}^{(0)}$, define the representation π_x of $C_c(\mathcal{G})$ in $\ell^2(\mathcal{G}_x)$ by

A

$$\forall \xi \in \ell^2(\mathcal{G}_x), \gamma \in \mathcal{G}_x, \ (\pi_x(f)\xi)(\gamma) = \sum_{s(\gamma_1)=s(\gamma)} f(\gamma\gamma_1^{-1})\xi(\gamma_1)$$

$C_r^*(\mathcal{G})$ is the completion of $C_c(\mathcal{G})$ with respect to the norm

$$\|f\|_r = \sup_{x\in X} \|\pi_x(f)\|$$

1. assuming that $\mathcal{G}$ is Hausdorff

Paterson, Khoshkam-Skandalis : If $\mathcal{G}_S$ is the groupoid associated to an inverse semigroup S we have

$$C^*(S) = C^*(\mathcal{G}_S), \;\; C^*_r(S) = C^*_r(\mathcal{G}_S).$$

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In case where P is a sub-semigroup of a group, we have therefore

$$C^*(P) = C^*(S(P)) = C^*(\mathcal{G}_{S(P)}) \twoheadrightarrow C^*_r(\mathcal{G}_{S(P)}) = C^*_r(S(P)) \overset{h}{\twoheadrightarrow} C^*_r(P) \overset{\kappa}{\hookrightarrow}$$

$$\overset{\kappa}{\hookrightarrow} W(P, G).$$

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The groupoid approach in the study of semigroup C^*-algebras was initiated by Muhly-Renault, carried on by Nica, Renault and many others, recently Xin Li, Sundar,...

So the weak containment problem for P or S is closely related to the same problem for étale groupoids, and similarly for the study of nuclearity.

AMENABILITY for GROUPOIDS

There are many equivalent definitions, as in the group case. We only mention one of them.

A function $k : \mathcal{G} \to \mathbb{C}$ is said to be **positive definite** if for every $x \in X = \mathcal{G}^{(0)}$ and every finite subset F of $\mathcal{G}^x$ the matrix $[k(\gamma^{-1}\gamma']_{\gamma,\gamma' \in F}$ is positive definite.

$\mathcal{G}$ is **amenable** iff there exists a net (k_i) of continuous positive definite functions in $C_c(\mathcal{G})$ such that

- $k_i^{(0)} \leq 1$, where $k_i^{(0)}$ is the restriction of k_i to $\mathcal{G}^{(0)}$;
- $\lim_i k_i = 1$ uniformly on every compact subset of $\mathcal{G}$.

Let $\mathcal{G}$ be an étale groupoid and let us consider the following conditions :

(1) $\mathcal{G}$ is amenable

(2) $C_r^*(\mathcal{G})$ is nuclear

(3) $C^*(\mathcal{G}) = C_r^*(\mathcal{G})$ (**weak containment property**)

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We left unsolved the problem of whether the equality $C^*(\mathcal{G}) = C_r^*(\mathcal{G})$ implies the amenability of $\mathcal{G}$.

A possible obstruction for (3) $\Rightarrow$ (1) : let F be an invariant (that is $r(\gamma) \in F \Leftrightarrow s(\gamma) \in F$) closed subspace of $X = \mathcal{G}^{(0)}$ and let $\mathcal{G}(F) = r^{-1}(F)$ be the restriction of $\mathcal{G}$ to F.

If (3) $\Rightarrow$ (1), then the weak containment property for $\mathcal{G}$ must imply the same property for $\mathcal{G}(F)$ for every such F, since amenability is preserved under restriction.

Let F be an invariant closed subset of $X = \mathcal{G}^{(0)}$ and set $U = X \setminus F$. The following diagram is commutative

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C^*(\mathcal{G}(U)) & \longrightarrow & C^*(\mathcal{G}) & \longrightarrow & C^*(\mathcal{G}(F)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow p & & \downarrow p_F & & \\
0 & \longrightarrow & C_r^*(\mathcal{G}(U)) & \longrightarrow & C_r^*(\mathcal{G}) & \longrightarrow & C_r^*(\mathcal{G}(F)) & \longrightarrow & 0
\end{array}
$$

where the first line is exact. Assume that p is injective. Then the second line is also exact if and only if p_F is injective.

Let $\mathcal{G}$ be an exact étale groupoid which is a bundle of groups. Then $\mathcal{G}$ has the weak containment property if and only if $\mathcal{G}$ is amenable.

Indeed, for $x \in X = \mathcal{G}^{(0)}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^*(\mathcal{G}(X \setminus \{x\})) & \longrightarrow & C^*(\mathcal{G}) & \longrightarrow & C^*(\mathcal{G}(x)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow p & & \downarrow p_x & & \\
0 & \longrightarrow & C_r^*(\mathcal{G}(X \setminus \{x\})) & \longrightarrow & C_r^*(\mathcal{G}) & \longrightarrow & C_r^*(\mathcal{G}(x)) & \longrightarrow & 0
\end{array}$$

p_x is injective whenever p is injective and the second line is exact.

Another example where weak containment implies amenability :

Let Γ be a discrete group. Then $C^*(\partial\Gamma \rtimes \Gamma) = C_r^*(\partial\Gamma \rtimes \Gamma)$ iff the groupoid $\partial\Gamma \rtimes \Gamma$ is amenable.

Another example where weak containment implies amenability :

Let Γ be a discrete group. Then $C^*(\partial\Gamma \rtimes \Gamma) = C^*_r(\partial\Gamma \rtimes \Gamma)$ iff the groupoid $\partial\Gamma \rtimes \Gamma$ is amenable.

Indeed, if $C^*(\partial\Gamma \rtimes \Gamma) = C^*_r(\partial\Gamma \rtimes \Gamma)$, using the commutativity of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(\Gamma \rtimes \Gamma) & \longrightarrow & C^*(\beta\Gamma \rtimes \Gamma) & \longrightarrow & C^*(\partial\Gamma \rtimes \Gamma) & & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & C^*_r(\Gamma \rtimes \Gamma) & \longrightarrow & C^*_r(\beta\Gamma \rtimes \Gamma) & \longrightarrow & C^*_r(\partial\Gamma \rtimes \Gamma) & \longrightarrow & 0 \end{array}$$

we see that the bottom line is exact.

Then a result of Roe-Willett (2013) states that this property implies that the metric space Γ has Yu's property A.

The first example of non exactness of such sequence

$$0 \longrightarrow C_r^*(\mathcal{G}(U)) \longrightarrow C_r^*(\mathcal{G}) \longrightarrow C_r^*(\mathcal{G}(F)) \longrightarrow 0$$

of reduced C^*-algebras is due to Skandalis (1991).

Since then, looking for counterexamples to Baum-Connes conjectures, many examples of non exact Hausdorff étale groupoids have been constructed.

GROUPOIDS ASSOCIATED WITH METRIC SPACES

Given a countable metric space X with bounded geometry,
Skandalis-Tu-Yu have constructed an étale Hausdorff principal groupoid $G(X)$ (which is $\beta\Gamma \rtimes \Gamma$ when $X = |\Gamma|$), whose reduced C^*-algebra is the uniform Roe C^*-algebra $C_u^*(X)$.

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One has the following equivalent properties :

(1) X has Yu's property A ;

(2) the groupoid $G(X)$ is amenable ;

(3) $C_r^*(G(X)) = C_u^*(X)$ is nuclear ;

(4) $C_r^*(G(X)) = C_u^*(X)$ is exact.

That (1) $\Leftrightarrow$ (2) is due to Skandalis-Tu-Yu, and the equivalence with (4) is a recent result of Sako.

Example : box spaces

Let Γ be a finitely generated residually finite group and let $\Gamma = N_0 \supset N_1 \cdots \supset N_k \supset \cdots$ be a decreasing sequence of finite index normal subgroups with $\cap_k N_k = \{e\}$.[2]

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Let Γ be a finitely generated residually finite group and let $\Gamma = N_0 \supset N_1 \cdots \supset N_k \supset \cdots$ be a decreasing sequence of finite index normal subgroups with $\cap_k N_k = \{e\}$. [2]

Set $X = \sqcup_k \Gamma/N_k$ endowed with the following metric : on the finite groupe $\Gamma_k := \Gamma/N_k$ it is the distance function associated with a generating set of Γ, and the distance between Γ/N_k and Γ/N_k tends to infinity when k, l tend to infinity.

Higson proved that when Γ has Kazhdan property T, the C^*-algebra $C_u^*(X)$ is not exact and showed that X provides a counterexample to the coarse Baum-Connes conjecture.

The representation $\pi = \oplus_{k\in\mathbb{N}}\lambda_k$ (where λ_k is the quasi-regular representation of Γ in $\ell^2(\Gamma_k)$) in $\ell^2(X) = \oplus_{k\in\mathbb{N}}\ell^2(\Gamma_k)$ plays a crucial role.

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The representation $\pi = \oplus_{k \in \mathbb{N}} \lambda_k$ (where λ_k is the quasi-regular representation of Γ in $\ell^2(\Gamma_k)$) into $\ell^2(X) = \oplus_{k \in \mathbb{N}} \ell^2(\Gamma_k)$ plays a crucial role. Indeed we have

$$C^*_\pi(\Gamma) := \pi(C^*(\Gamma)) \subset C^*_u(X) \subset \mathcal{B}(\ell^2(X)),$$

and $C^*_\pi(\Gamma)$ is not exact when Γ has property T.

In fact $C^*_\pi(\Gamma)$ is exact iff the group Γ is amenable.

Example : Higson-Lafforgue-Skandalis groupoids

Let Γ be a residually finite group and $(N_k)_{k\in\mathbb{N}}$ an approximating sequence as above. We set $\Gamma_k = \Gamma/N_k$ and $\Gamma_\infty = \Gamma$ and denote by $q_k : \Gamma \to \Gamma_k$ the quotient map. Let $\widehat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the Alexandroff compactification of $\mathbb{N}$. Let $\mathcal{G}$ be the quotient of $\widehat{\mathbb{N}} \times \Gamma$ by the equivalence relation

$$(k, s) \sim (l, t) \quad \text{if} \quad k = l \quad \text{and} \quad q_k(s) = q_k(t).$$

Equipped with the quotient topology, $\mathcal{G}$ is an étale Hausdorff groupoid, a bundle of groups, whose fibre (i.e. isotropy) at k is $\mathcal{G}(k) = \Gamma_k$.

◂ F

For $f \in C_c(\mathcal{G})$, recall that $\pi_k(f)$ acts on $\ell^2(\Gamma_k)$ and we have

$$\pi_k(C_r^*(\mathcal{G})) = \lambda_k(C_r^*(\Gamma))$$

where λ_k is the quasi-regular representation of Γ in $\ell^2(\Gamma_k)$.

Higson-Lafforgue-Skandalis groupoids

$C^*_r(\mathcal{G})$ is a lower semicontinuous field of C^*-algebras over $\widehat{\mathbb{N}}$ with fibre $C^*_r(\Gamma_k)$ at $k \in \widehat{\mathbb{N}}$. We have $C^*_r(\mathcal{G}(\mathbb{N})) = \bigoplus_{k \in \mathbb{N}} C^*_r(\Gamma_k)$.

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Let $\pi = \oplus_{k\in\mathbb{N}}\lambda_k$. Higson-Lafforgue-Skandalis have proved that whenever the trivial representation of Γ is isolated in the support of π (e.g. if Γ has Kazhdan property T) then, not only

$$0 \longrightarrow C_r^*(\mathcal{G}(\mathbb{N})) \longrightarrow C_r^*(\mathcal{G}) \longrightarrow C_r^*(\mathcal{G}(\infty)) = C_r^*(\Gamma) \longrightarrow 0$$

is not exact in the middle, but also

$$K_0\big(C_r^*(\mathcal{G}(\mathbb{N}))\big) \longrightarrow K_0\big(C_r^*(\mathcal{G})\big) \longrightarrow K_0\big(C_r^*(\mathcal{G}(\infty))\big)$$

is not exact in the middle.

The restriction map $f \in C_c(\mathcal{G}) \mapsto f_{|_{\mathcal{G}(\infty)}}$ from $C_c(\mathcal{G})$ onto $\mathbb{C}[\Gamma]$ induces an isomorphism from $C_r^*(\mathcal{G})/C_r^*(\mathcal{G}(\mathbb{N}))$ onto $C_\pi^*(\Gamma) := \pi(C^*(\Gamma))$.

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So we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^*(\mathcal{G}(\mathbb{N})) & \longrightarrow & C^*(\mathcal{G}) & \longrightarrow & C^*(\Gamma) & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_\pi} & & \\
0 & \longrightarrow & C_r^*(\mathcal{G}(\mathbb{N})) & \longrightarrow & C_r^*(\mathcal{G}) & \longrightarrow & C_\pi^*(\Gamma) & \longrightarrow & 0 \\
 & & \Vert & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_r^*(\mathcal{G}(\mathbb{N})) & \longrightarrow & C_r^*(\mathcal{G}) & \longrightarrow & C_r^*(\Gamma) & \longrightarrow & 0
\end{array}$$

where the two first lines are exact.
In particular we see that p is injective (i.e. $\mathcal{G}$ has the weak containment property) iff p_π is injective, that is

$$\forall a \in C^*(\Gamma),\ \|a\|_{C^*(\Gamma)} = \sup_k \|\lambda_k(a)\|.$$

- $\mathcal{G}$ has the weak containment property iff

$$\forall a \in C^*(\Gamma),\ \|a\|_{C^*(\Gamma)} = \sup_k \|\lambda_k(a)\| \tag{1}$$

- $\mathcal{G}$ is amenable iff the group Γ is amenable

⤳ $\mathcal{G}$ has the weak containment property iff

$$\forall a \in C^*(\Gamma),\ \|a\|_{C^*(\Gamma)} = \sup_k \|\lambda_k(a)\| \tag{1}$$

⤳ $\mathcal{G}$ is amenable iff the group Γ is amenable

Willett has recently (2015) provided an example of a non amenable group Γ (namely $\Gamma = \mathbb{F}_2$) and an approximating sequence $(N_k)_{k\geq 0}$ of subgroups for which (1) holds, that is the irreducible representations of Γ that factors through some N_k are dense in the dual of Γ.

Note that in this example the second line of the diagram below is not exact in the middle :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^*(\mathcal{G}(\mathbb{N})) & \longrightarrow & C^*(\mathcal{G}) & \longrightarrow & C^*(\mathbb{F}_2) & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow & & \\
0 & \longrightarrow & C_r^*(\mathcal{G}(\mathbb{N})) & \longrightarrow & C_r^*(\mathcal{G}) & \longrightarrow & C_r^*(\mathbb{F}_2) & \longrightarrow & 0
\end{array}$$

A positive result :

Matsumura proved (2012) that if Γ is an **exact** discrete group acting by homeomorphisms on a **compact** space X, then the weak containment property of the transformation groupoid $\mathcal{G} = X \rtimes \Gamma$ implies the nuclearity of $C_r^*(\mathcal{G})$. His method consists in showing that the embedding of $C_r^*(\mathcal{G})$ in its bidual is nuclear.

It is likely that this fact extends to the case of étale groupoids satisfying an appropriate definition of exactness.

EXACT DISCRETE GROUPS

Let us recall that for a discrete group Γ the following conditions are equivalent :

(1) Γ acts amenably on a compact space ;

(2) Γ is exact in the sense of Kirchberg-Wassermann, that is, for every exact sequence of Γ-C^*-algebras,

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

the corresponding sequence

$$0 \longrightarrow I \rtimes_r \Gamma \longrightarrow A \rtimes_r \Gamma \longrightarrow B \rtimes_r \Gamma \longrightarrow 0$$

of reduced crossed product C^*-algebras is exact ;

(3) the reduced C^*-algebra $C_r^*(\Gamma)$ is exact.

What about exact étale groupoids ?

For an étale groupoid $\mathcal{G}$, we may in the same way consider the following conditions :

(1) $\mathcal{G}$ acts amenably on a fibre space $Y \xrightarrow{p} \mathcal{G}^{(0)}$ such that p is proper ;

(2) $\mathcal{G}$ is exact in the sense of Kirchberg-Wassermann ;

(3) the reduced C^*-algebra $C_r^*(\mathcal{G})$ is exact.

We have (1) $\Rightarrow$ (2) $\Rightarrow$ (3), but except in particular cases (for instance if $\mathcal{G}$ is Morita equivalent to a transformation groupoid) I don't know much about the converse.

For instance, if $\mathcal{G} = X \rtimes \Gamma$ for a partial action of an exact group Γ, then $C_r^*(\mathcal{G})$ is exact, but I don't know whether $\mathcal{G}$ is exact in the sense of Kirchberg-Wassermann.