Uniform Companions for Large Differential Field Expansions of Characteristic 0

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Poster Abstract

This poster presents a method of extending a model companion (resp. completion) T_1 of a theory T_0 of fields of char. 0 to a model companion (resp. completion) of $T_0 \cup DF_N$ where DF_N is the theory of differential fields in $N \in \mathbb{N}$ commuting derivations.

Introndution

Our method shall apply to any theory T_1 of a language $\mathcal{L}$ extending $\mathcal{L}_{Ri} := \{+, -, \cdot, 0, 1\}$ for where one can define the following kind of topology with existential formulae

Definable Large Topology in T_1

In each model $\mathcal{M} \models T_1$, expanding a field K say, there exists an existentially definable basis (possibly with parameters) in $\mathcal{L}$ for a topology τ on K such that

1. Every $\mathcal{L}$-atomic set in a substructure $\mathcal{A}$ of $\mathcal{M}$ is locally (w.r.t to τ) a projection of a constructible set defined over $\mathcal{A}$.
2. If an irreducible K-constructible set V has a regular K-rational point that also lies in some open set $U \in \tau$, then the set of K-rational points in U in $\mathcal{M}$ is Zariski-dense in V.

With this we show that (Notation: $\bar{\partial} := (\partial_1, \ldots, \partial_N)$ is a tuple of unary function symbols)

Main Theorem

For each set $\mathfrak{B}$ of $\exists$-$\mathcal{L}$-formulae there exists a $\mathcal{L}(\bar{\partial})$-theory UC_N s.t. for all T_0, T_1 as above, if $\mathfrak{B}$ defines a basis for a large topology in T_1 then $T_1 \cup UC_N$ is the model companion (resp. completion) of $T_0 \cup DF_N$. Furthermore, if T_1 has quantifier elimination or NIP then so does $T_1 \cup UC_N$

Since the theory UC_N remains fixed for a fixed definition schema $\mathfrak{B}$ for a basis of a topology we call UC_N the **uniform companion for fields with such a topology**. This result generalises that of M. Tressl (cf. [4]), N. Guzy (cf. [2]) and C. Rivière (cf. [3]).

The Theory UC_N

Intuition Behind Axioms

Reduce the solvability of systems of differential equations in τ-open neighbourhoods TO of systems of algebraic eqns in τ-open neighbourhoods.

The systems of diff. eqns we consider

Let $\mathcal{M}$ be a model of $T_1 \cup DF_N$ and expand a differential field K. Condition .1 of large topologies means we only need to consider the following systems of diff. eqns

$$f_1(\bar{x}) = \ldots = f_m(\bar{x}) = 0 \wedge H(f_1, \ldots, f_m)(\bar{x}) \neq 0, \bar{x} \in U$$

- $f_1, \ldots, f_m \in K\{\bar{x}\}$ form a characteristic set of some differential prime ideal of $K\{\bar{x}\}$ ($\bar{x} := (x_1, \ldots, x_n)$).
- $H(f_1, \ldots, f_m)$ is the multiple of the separants and initials of each $f_1, \ldots, f_m$.
- $U \subseteq K^n$ a basic open set.

Call such a system a **topological prime system**.

A solution of such a system - a tuple is some differential field and $\mathcal{L}$-structure extension $\mathcal{N}$ of $\mathcal{M}$ that solves the system $f_1(\bar{x}) = \ldots = f_m(\bar{x}) = 0, H(f_1, \ldots, f_m) \neq 0$ and is in the "extension" of U.

For any differential polynomial $f \in K\{\bar{x}\}$ let f^* be the **non-differential** polynomial obtained by replacing the variables $\partial_1^{i_1} \ldots \partial_N^{i_N} x_j$ in f with a standard variable.

Proposition 1

If $\{f_1, \ldots, f_m\} \subseteq K\{\bar{x}\}$ is a characteristic set for some differential prime ideal of $K\{\bar{x}\}$, then the system of **standard** polynomial equations

$$f_1^* = 0, \ldots = f_m^* = 0, y \cdot H(f_1, \ldots, f_m)^* - 1 = 0 \qquad (1)$$

defines an irreducible constructible set $V_{f_1,\ldots,f_m}$ in the algebraic closure of K.

Axioms of UC_N

For any model $\mathcal{M} \models UC_N$, $\mathcal{M}$ is a differential field and

For every topological prime system

$$f_1(\bar{x}) = \ldots = f_m(\bar{x}) = 0, H(f_1, \ldots, f_m)(\bar{x}) \neq 0, \bar{x} \in U$$

if $V_{f_1,\ldots,f_m}$ has a regular K-rational point $\bar{a}$ such that $(a_1, \ldots, a_n) \in U$ then the system is solvable in $\mathcal{M}$

Theorem I

If for some $\mathcal{M}, \mathcal{N} \models T_1 \cup UC_N$ such that $\mathcal{M} \equiv_{\mathcal{L},\exists,A} \mathcal{N}$ where A is a common differential subring, then $\mathcal{M} \equiv_{\mathcal{L}(\bar{\partial}),\exists,A} \mathcal{N}$.

Consequence of Condition 2. of Large Topologies

2. of the definition of large topologies (left) gives the following important consequence-

Proposition 2

Let $\mathcal{M} \models T_1$ expanding some field K. If for each irreducible affine K-variety V (of arbitrary dimension) with a regular K-rational point in some $U \in \tau$, then there exists some elementary $\mathcal{L}$-extension $\mathcal{N} \succ \mathcal{M}$ containing a generic point $\bar{\alpha}$ of V such that α is in the extension of U in $\mathcal{N}$.

By applying this to the varieties $V_{f_1,\ldots,f_m}$ in proposition 1, we can obtain the following-

Theorem II

For any $\mathcal{M} \models T_1 \cup DF_N$ then there exists some $\mathcal{L}(\bar{\partial})$-extension $\mathcal{N}$ that is also a differential field extension such that $\mathcal{M} \prec_{\mathcal{L}} \mathcal{N}$ and $\mathcal{N} \models UC_N$.

In fact, Theorems I and II together actually give the Main Theorem.

Applications: Valued and Ordered Fields

Proposition 3

Theories of the following fields have a large existentially definable topology

1. Henselian valued fields (in $\mathcal{L}_{Ri}(|)$ and $\mathcal{L}_{Ri}(|, \{P_n\}_{n\in\mathbb{N}})$).
2. PRC$_e$ field (in the langauge $\mathcal{L}_{Ri}(<_1, \ldots, <_e)$)
3. Existentially closed fields with several valuations and orderings (in a lang. extending $\mathcal{L}_{Ri}$ by the orders, valuations and, if needs be, predicates $\{P_n\}_{n\in\mathbb{N}}$).

In each of these cases, by picking $\mathfrak{B}$ in Main Theorem to be the definition schema of the basis of the topology we obtain an uniform companion for the given kind of topological fields.

Future Work and Questions

1. Preservation of other model theoretic properties UC_N (e.g. stability, rosiness).
2. Can the methods be adapted for theories of expansions of fields by functions (e.g. theories of difference fields or exponential fields)?
3. How do models of UC_N behave under taking algebraic extensions?
4. Darnière showed that there exists model complete theories of fields extended by a subring and a radical relation on the ring (cf. [1]). For models of such a theory, does the radical relation induce a large topology?

References

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