Some properties of the differential ideal $[x^m]$.

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Abstract

The differential ideal $[x^m]$, $m \in \mathbb{N}$, first appeared in Levi's paper [1]. He used combinatorial properties of $[x^m]$ in the proof of low power theorem. In his book [2] Ritt posed the question: what is the minimal k such that $\left(x^{(s)}\right)^k \in [x^m]$.

We present an interesting method for investigation this ideal. We construct a monomorphism from $k\{x\}/[x^m]$ to an exterior algebra equipped with a derivation. The monomorphism allows to prove several facts, for example:

- $\left(x^{(s)}\right)^k \in [x^m]$ if and only if $k > (m-1)(s+1)$;
- if $f' \notin [x^m]$, then $f \notin [x^m]$;
- algebra $k\{x\}/[x^m]$ is prime.

We also provide an analogous construction for the ideal $\left[x^2, (x')^2, \ldots, \left(x^{(k)}\right)^2\right]$

Definitions

From now on, we will always assume that k is a field of characteristic zero.

We denote a differential polynomial algebra in one indeterminate by $k\{x\}$. More precisely, $k\{x\}$ is a polynomial algebra $k[x_0, x_1, x_2, \ldots]$ in a sequence of algebraic independent indeterminates ($x = x_0, x_1, x_2, \ldots$) endowed with a derivation (i.e. k-linear operator satisfying the Leibniz law) such that $x_n' = x_{n+1}$. An ideal I is called *a differential ideal* if $I' \subset I$.

A differential algebra A is said to be *prime* if $IJ \neq 0$ for any nonzero ideals $I, J \subset A$.

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The monomorphism

Let us consider an infinite dimensional vector space V_m with a basis consisting of ξ_i^k and η_i^k ($k = 0, \ldots, m-2$, $i \in \mathbb{Z}_{\geqslant 0}$). We denote a Grassmann algebra of V_m by $\Lambda(V_m)$ and denote its even and odd components by $\Lambda_0(V_m)$ and $\Lambda_1(V_m)$, respectively. Let us equip $\Lambda(V_m)$ with a derivation (not superderivation) using the formulas

$$(\xi_i^k)' = \xi_{i+1}^k \text{ and } (\eta_i^k)' = \eta_{i+1}^k$$

i.e. the derivation increments the subscript. Obviously, $\Lambda_0(V_m)$ is a differential subalgebra in $\Lambda(V_m)$. Let us denote the image of x in $k\{x\}/[x^m]$ by $\bar{x}$.

Theorem

Let us define a homomorphism of differential algebras

$$\varphi_m \colon k\{x\}/[x^m] \to \Lambda(V_m)$$

by the rule

$$\varphi_m(\bar{x}) = \xi_0^0 \wedge \eta_0^0 + \ldots + \xi_0^{m-2} \wedge \eta_0^{m-2}.$$

In the special case where $m = 2$ we have

$$\varphi_2(\bar{x}) = \xi_0 \wedge \eta_0.$$

Then, φ_m is a monomorphism.

Examples

Example 1. Let us prove that $xx_3^2 \notin [x^2]$:

$$\begin{aligned} &\varphi_2(\bar{x}\bar{x}_3^2) = \\ &= \xi_0\wedge\eta_0\wedge(\xi_3\wedge\eta_0+3\xi_2\wedge\eta_1+3\xi_1\wedge\eta_2+\xi_0\wedge\eta_3)^2 = \\ &= \xi_0 \wedge \eta_0 \wedge (3\xi_2 \wedge \eta_1 + 3\xi_1 \wedge \eta_2)^2 = \\ &= 18\xi_0 \wedge \eta_0 \wedge \xi_2 \wedge \eta_1 \wedge \xi_1 \wedge \eta_2 \neq 0 \end{aligned}$$

Example 2. Let us prove that $x_1^5 \in [x^3]$.

$$\varphi_3(\bar{x}_1^5) = (\xi_1^0\wedge\eta_0^0+\xi_0^0\wedge\eta_1^0+\xi_1^1\wedge\eta_0^1+\xi_0^1\wedge\eta_1^1)^5$$

Indeed, expanding brackets we obtain a homogeneous element of degree 10 involving only 8 different variables. Thus, $\varphi_3(x_1^5) = 0$, so $x_1^5 \in [x^3]$.

Applications of the monomorphism

Theorem

- $x_s^k \in [x^m]$ if and only if $k > (m-1)(s+1)$;
- if $f' \notin [x^m]$, then $f \notin [x^m]$;
- algebra $k\{x\}/[x^m]$ is prime.

Thus, the subalgebra consisting of all elements of $k\{x\}/[x^m]$ with zero constant term provides an example of prime differential nilalgebra.

Theorem

Dimension of the subalgebra of $k\{x\}/[x^m]$ generated by $x, x_1, \ldots, x_N$ equals m^{N+1}.

Possible generalizations

We define a homomorphism of differential algebras

$$\varphi_{2,s} \colon k\{x\}/\left[x, x_1^2, \ldots, x_s^2\right] \to \Lambda(V_{2+s})$$

by the rule

$$\varphi_{2,s}(\bar{x}) = \xi_0^0 \wedge \eta_0^0 \wedge \ldots \wedge \xi_0^s \wedge \eta_0^s$$

Then, $\varphi_{2,s}$ turns out to be a monomorphism.

Homomorphisms φ_m and $\varphi_{2,s}$ can be generalized in an essential way, but we do not know an appropriate set of generators for the kernel of the resulting homomorphism.

References

[1] Levi H., *On the structure of differential polynomials and their theory of ideals,* Trans. AMS, vol.51, 532-568, 1942.

[2] Ritt J.F., *Differential Algebra,* volume XXXIII of Colloquium Publications. New York, American Mathematical Society, 1950.

[3] Pogudin G.A., *A prime differential nilalgebra exists.*, http://arxiv.org/abs/1409.3847.