Mann Property and Interpretable Groups

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Mann Property

Let K be an algebraically closed field, the field $\mathbb{F}$ its prime field and G be an infinite multiplicative subgroup of $K^{\times}$. A fundamental notion about multiplicative groups is the *Mann Property*. To define this property, consider an equation

$$a_1 x_1 + ... + a_n x_n = 1 \quad (1)$$

with $n \geq 2$ and $a_i \in \mathbb{F}$. A solution $(g_1, ..., g_n)$ of this equation is called non-degenerate if for every non-empty subset I of $\{1, 2., , , .n\}$, the sum $\sum_{i \in I} a_i g_i$ is not zero. We say that G has *the Mann property* if every such equation (1) has only finitely many non-degenerate solutions in G.
Now fix K and G with the Mann property as above. By the pair (K, G), we mean the structure $(K, G, +, -, \cdot, 0, 1)$. So our language is $L(U) = \{+, -, \cdot, 0, 1, U\}$ where U is a unary relation whose interpretation in K is G.

Examples

- In (M), H. Mann showed that the set of complex roots of unity μ has the Mann property.
- In the 1980's, H. Mann's result was generalized and it was proven that any multiplicative group of finite rank (note that μ has rank 0) in any field of characteristic zero has the Mann property. For instance $2^{\mathbb{Z}}$ in $\mathbb{C}$ has the Mann property.

First Motivation for the Mann Property via Model Theory

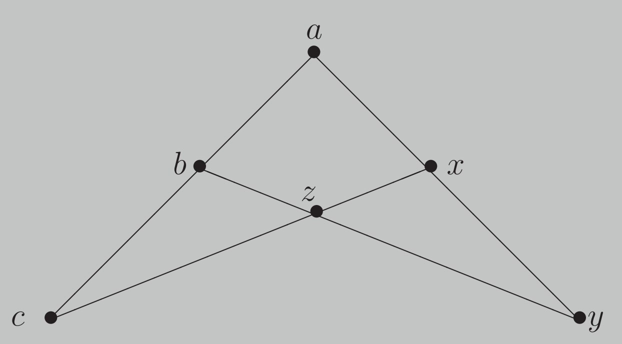
B. Zilber was the first one who considired the Mann property in model theory. In 1990, he studied the model theory of the pair $(\mathbb{C}, \mu)$. In his unpublished note, B. Zilber (Z) showed that the pair $(\mathbb{C}, \mu)$ is ω-stable, using as a main tool the result of H. Mann (M). This motivated the study of the pair (K, G) model theoretically with the aim of understanding its stability theoretic properties and definable sets and interpretable groups in it.

Stability Results

More generally, the model theory of the pair (K, G) was first studied in the paper of L. van den Dries and A. Günaydın (DG). Among other things, an axiomatization of the theory of (K, G) was obtained by adding constants to denote the collection of non-degenerate solutions of the equation (1). Moreover, they gave a relative quantifier elimination. L. van den Dries and A. Günaydın (DG) generalized B. Zilber's result to (K, G), where K is an algebraically closed and G has the Mann property, that is to say, the theory of (K, G) is stable and if G is superstable (ω-stable) in the pure group language then so is the pair (K, G).

Group Configuration

Let T be a stable theory and $\mathcal{U}$ its sufficiently saturated model. By a group configuration over A we mean 6-tuple of points (possibly infinite) (in $\mathcal{U}^{eq}$) (a, b, c, x, y, z) such that



- any triple of non-collinear points are A-independent,
- $acl^{eq}(A, a, b) = acl^{eq}(A, a, c) = acl^{eq}(A, b, c)$,
- x and y are interalgebraic over Aa, the elements y and z are interalgebraic over Ab, the elements z and x are interalgebraic over Ac,
- $a \in acl^{eq}(x, y, A)$, $b \in acl^{eq}(y, z, A)$ and $c \in acl^{eq}(x, z, A)$.

Group Configuration Theorem, E. Hrushovski (H)

Let T be a stable theory and $\mathcal{U}$ its sufficiently saturated model. Suppose $M \subset \mathcal{U}$ to be a $|T|^{+}$-saturated model of T, and suppose (a, b, c, x, y, z) is a group configuration over M. Then there is a $*$-definable group G in $\mathcal{U}^{eq}$ over M, and there are elements a', b', c', x', y', z' of G which form a group configuration, each generic over M, such that the element a is interalgebraic with a' over M and the same holds for the other elements. Moreover, we have $a'x' = y'$, $b'y' = z'$, $c'x' = z'$ and $b'a' = c'$.

Tools for the Characterization of Interpretable Groups

Characterization of the model-theoretic algebraic closure and the independence in (K, G) enable us to characterize definable groups in (K, G) up to isogeny, in terms of definable and interpretable groups in K and G. The proof entails the group configuration theorem. We follow the approach of (BMP), where T. Blossier and A. Martin-Pizarro characterized interpretable groups in pairs of proper extension of algebraically closed fields using a result of A. Pillay (P). In the case where G is divisible, we can characterize interpretable groups in (K, G) in terms of definable groups in (K, G).

Characterization of Definable Groups

Let K be an algebraically closed field and G be an infinite multiplicative subgroup of $K^{\times}$ with the Mann property. Any type-definable group in (K, G) is isogenous to a subgroup of an algebraic group. Moreover any type-definable group is, up to isogeny, an extension of a type-interpretable group in G by an algebraic group.

Characterization of Interpretable Groups

Let K be an algebraically closed field and G be a divisible multiplicative subgroup of $K^{\times}$ with the Mann property. Every interpretable group H in (K, G) is, up to isogeny, an extension of an interpretable abelian group in G by an interpretable group N, which is a quotient of an algebraic group V by a subgroup N_1, which is an interpretable abelian group in G.

References

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