Summer School in Tame Geometry

University of Konstanz, July 18-23 2016

Tutorials on topics in real geometry, o-minimal geometry and tame geometry, given by Philipp Hieronymi (University of Illinois, Urbana-Champaign), Tobias Kaiser (University of Passau), Margarita Otero (Universidad Autónoma de Madrid), Ya'acov Peterzil (University of Haifa), Daniel Plaumann (University of Konstanz), Margaret Thomas (University of Konstanz), as well as survey lectures on surrounding topics.

Organisers: Pantelis Eleftheriou (Konstanz), Salma Kuhlmann (Konstanz), Daniel Plaumann (Konstanz), Jonathan Pila (Oxford), Margaret Thomas (Konstanz)

Effective Pila-Wilkie bounds for restricted Pfaffian surfaces

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Ordered Algebraic Structures and Related Topics
CIRM Luminy
October 13th 2015

Pila–Wilkie Theorem

Let $X \subseteq \mathbb{R}^n$ be definable in an o-minimal expansion of the ordered field of real numbers. Let X^{trans} be the set obtained by removing all infinite, connected semi-algebraic subsets from X.
Let $\varepsilon > 0$.
There exists $c = c(X, \varepsilon) > 0$ such that, for all $H \geq 1$, X^{trans} contains at most $c.H^{\varepsilon}$ rational points of height at most H.

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$\text{ht}(a/b) := \max\{|a|, |b|\}$, and $\text{ht}((q_1,\ldots,q_n)) := \max_{i=1,\ldots,n}\{\text{ht}(q_i)\}$.

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We want that the constant $c(X,\varepsilon)$ can be effectively computed.

Theorem (Jones-T. (≥ 2015))

Suppose that $f : (0,1)^2 \to (0,1)$ *is implicitly defined from restricted Pfaffian functions with complexity at most* B*.*
Let $\varepsilon > 0$*. There is an effective* $c = c(B,\varepsilon) > 0$ *such that, for all* $H \geq 1$,

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Restricted Pfaffian surfaces

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- They are analytic and o-minimal ($\mathbb{R}_{\text{Pfaff}}$, the expansion of the real ordered field by all Pfaffian functions, is o-minimal).
- They include $\exp$ on $\mathbb{R}$, $\log$ on $(0,\infty)$, $\sin$ on $(0,\pi)$, ...

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(Khovanskii 1980) For Pfaffian functions g, we have effective bounds on, say, the number of connected components of zero sets $V(g), V(g'), \ldots$, which depend only on the number of variables of g, the number of functions r defined by the system defining g, and the degrees of the polynomials in the system.

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If $g:U \to \mathbb{R}^m$ is Pfaffian and analytic on an open set $U \supseteq [0,1]^n$, then the corresponding restricted function $\hat{g}:\mathbb{R}^n \to \mathbb{R}$ is given by

$$\hat{g}(\bar{x}) = \begin{cases} g(\bar{x}) & \text{if } \bar{x} \in [0,1]^n \\ 0 & \text{otherwise.} \end{cases}$$

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The real ordered field expanded by all restricted Pfaffian functions is o-minimal.

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A function $f: U \to \mathbb{R}$ (here $U \in \mathbb{R}^m$) is implicitly defined from restricted Pfaffian functions if there exist $n \geq 1$, restricted Pfaffian functions $p_1,\dots,p_n : \mathbb{R}^{m+n} \to \mathbb{R}$ and functions $f_2,\dots,f_n : U \to \mathbb{R}$ s.t.

$$p_1(\bar{x}, f(\bar{x}), f_2(\bar{x}), \dots, f_n(\bar{x})) = \dots = p_n(\bar{x}, f(\bar{x}), f_2(\bar{x}), \dots, f_n(\bar{x})) = 0$$
$$\det\left(\frac{\partial(p_1,\dots,p_n)}{\partial(y_1,\dots,y_n)}\right)(\bar{x}, f(x), f_2(\bar{x}), \dots, f_n(\bar{x})) \neq 0.$$

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If $p_1, \ldots, p_n$ have a common system of definition, then the implicit definition of f has a certain complexity given in terms of the number of functions defined by that system, and the degrees of the polynomials involved. Likewise, there are analagous effective bounds for these implicitly defined functions, given in the same terms.

Pila-Wilkie Main Lemma

Let $k < n$. For all $d \geq 1$, there exist effective $\widetilde{r}(k,n,d) \in \mathbb{N}$, $\widetilde{\varepsilon}(k,n,d), \tilde{C}(k,n,d) > 0$ such that $\widetilde{\varepsilon}(k,n,d) \to 0$ as $d \to \infty$ and if Φ is an $C^{\widetilde{r}}$-parameterization of $X \subseteq \mathbb{R}^n$ with $\dim(X) = k$, then $X(\mathbb{Q}, H)$ is contained in at most $\#\Phi.\tilde{C}.H^{\widetilde{\varepsilon}(k,n,d)}$ sets $V(P)$ with $\deg(P) \leq d$.

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Idea: fix $\varepsilon > 0$. Since $\widetilde{\varepsilon}(k,n,d) \to 0$ as $d \to \infty$, so we can (effectively) find d large enough that $\widetilde{\varepsilon}(k,n,d) < \varepsilon$.

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$\widetilde{r}(k,n,d), \tilde{C}(k,n,d)$ are effective in d and hence in ε.

- We need an effective $C^{\widetilde{r}}$-parameterization Φ of X such that $\#\Phi$ is bounded effectively in the "complexity of X";
- We need an effective estimate of $\#(X \cap V(P))^{\text{trans}}(\mathbb{Q}, H)$ which is $O(H^{\varepsilon})$, for polynomials P with $\deg(P) \leq d$.

Zero estimates

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The second is going to be a consequence, for surfaces implicitly defined from Pfaffian functions, either of earlier work (Jones-T. 2012), or using an argument analogous to that of the original proof of the Pila-Wilkie Theorem, but in the same spirit as what now follows for C^r-parameterization. Either way it uses the bounds mentioned earlier for the zero sets of functions implicitly defined from Pfaffian functions.

Pila-Wilkie Parameterization Theorem

Let $X \subseteq (0,1)^m$ be definable in an o-minimal expansion of a real closed field. For all $r \geq 1$, there exists a finite set Φ of C^r maps $\phi\colon (0,1)^k \to (0,1)^m$, where $\dim(X) = k$, such that

- $X = \bigcup_{\phi \in \Phi} \mathrm{Im}(\phi)$;
- for all $\phi \in \Phi$, $\left|\left|\phi^{(\alpha)}\right|\right| \leq 1$, for all $|\alpha| \leq r$.

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So, to avoid compactness, work uniformly in families throughout!

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Lemma

Let $F:(0,1)^2 \to (0,1)^n$ be implicitly defined of complexity at most B. There exist real numbers $0=\xi_0<\xi_1<\ldots<\xi_N<\xi_{N+1}=1$ and implicitly definable functions $\phi_{i,j}:C_i \to (0,1)$, where $C_i=(0,1)\times(\xi_i,\xi_{i+1})$, such that, for $y \in (\xi_i,\xi_{i+1})$, the functions $\phi_{i,0}(\cdot,y),\ldots,\phi_{i,M_i}(\cdot,y)$ form an r-parameterization of the graph of $F(\cdot,y)$. Moreover, $N,M_1,\ldots,M_N$ and the complexity of the implicit definitions of the $\phi_{i,j}$ are all bounded by an effective constant depending only on B and r.

Another remark: at a certain point it is necessary to take limits of parameterizations. We manage to avoid the use of effective model completeness for this step, which would make the bounds much worse, by employing an effective stratification due to Gabrielov and Vorobjov. (This also plays a role at various other points in the proof.)