Lebesgue integration of oscillating and subanalytic functions

Tamara Servi

(University of Paris 7)

(joint work with R. Cluckers, G. Comte, D. Miller, J.-P. Rolin)

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An application of o-minimality to an oscillating context

o-minimal context: globally subanalytic sets and functions (i.e. def. in $\mathbb{R}_{an}$)
Proviso. For the rest of the talk, *subanalytic* means "globally subanalytic".
Def. For $X \subseteq \mathbb{R}^m$ subanalytic, let $\mathcal{S}(X) := \{f : X \to \mathbb{R} \text{ subanalytic}\}$

Puiseux-Lojasiewicz. Let $f(y) \in \mathcal{S}(\mathbb{R})$. Then $\exists c > 0$ s.t. $\forall y > c$
$f(y) = ay^r H\left(y^{-\frac{1}{d}}\right)$, where $d \in \mathbb{N}, r \in \mathbb{Q}, a \in \mathbb{R}$ and $H(Y) \in \mathbb{R}\{Y\}^*$.

Subanalytic Preparation Theorem (Lion - Rolin). Let $f(\overline{x}, y) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R})$. Then there is a subanalytic cell decomposition of $\mathbb{R}^{n+1}$ such that on every cell of the form $\{(\overline{x}, y) : \ x \in X, \ y > c(\overline{x}) > 1\}$

$$f(\overline{x}, y) = a(\overline{x})\, y^r U(\overline{x}, y), \text{ where}$$

$$r \in \mathbb{Q}, a \in \mathcal{S}(X) \text{ and } U(\overline{x}, y) = H\left(y^{-\frac{1}{d}}\right), \text{ with } d \in \mathbb{N} \text{ and } H \in \mathcal{S}(X)\{Y\}^*.$$

General philosophy: presentation of f which is as simple as possible wrto a chosen variable y (possibly at the price of complicating the situation in $\overline{x}$).

- *Monomialization* respecting y (resolution of singularities):
  setting $y_1 = yU(\overline{x}, y)^{\frac{1}{r}}$, we have $f_1(\overline{x}, y_1) = a(\overline{x})\, y_1^r$.
- Useful to handle *logarithms* of subanalytic functions:
  $\log(f) = r \log y + \log(a(\overline{x})) + \log(U(\overline{x}, y))$.

... and van den Dries said: "now go and integrate!"
They did, and they saw that it was good. And so our story begins.

Motivation and background

Oscillatory integrals of the 1^{st} kind. $x \in \mathbb{R}$, $\overline{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$

$$\mathcal{I}(x) = \int_{\mathbb{R}^n} e^{\mathrm{i}x\varphi(\overline{y})} \psi(\overline{y})\, \mathrm{d}\overline{y}, \text{ where:}$$

- the *phase* φ is analytic, $0 \in \mathbb{R}^n$ is an isolated singular point of φ;
- the *amplitude* ψ is $\mathcal{C}^\infty$ with support a compact nbd of 0.

These objects are studied in optical physics, acoustics and number theory.

AIM. To study the behaviour of $\mathcal{I}(x)$ when $x \to \infty$.

$n = 1$ $\quad \mathcal{I}(x) \sim e^{\mathrm{i}x\varphi(0)} \sum_{j \in \mathbb{N}} a_j(\psi)\, x^{-\frac{j}{N(\varphi)}} \quad a_j(\psi) \in \mathbb{R}$, $N(\varphi) \in \mathbb{N}$ fixed.

$n > 1$ reduce to the case $n = 1$ by *monomializing* the phase (res. of sing.).

Example. $n = 2$, $\quad \mathcal{I}(x) = \iint e^{\mathrm{i}x y_1^a y_2^b} \psi(y_1, y_2)\, \mathrm{d}y_1 \mathrm{d}y_2$

$\Phi : (y_1, y_2) \mapsto (Y_1, Y_2) = \left(y_1, y_1^a y_2^b\right), \quad \tilde{\psi} = \psi \circ \Phi^{-1} \cdot \mathrm{Jac}\Phi^{-1}$

$\mathcal{I}(x) = \int \left(\int e^{\mathrm{i}xY_2} \tilde{\psi}(Y_1, Y_2)\, \mathrm{d}Y_2 \right) \mathrm{d}Y_1 \quad$ (Fubini)

Monomializing the phase, using Fubini and the case $n = 1$, one proves:

$$\mathcal{I}(x) \sim e^{\mathrm{i}x\varphi(0)} \sum_q \sum_{k=0}^{n-1} a_{q,k}(\psi)\, x^q (\log x)^k .$$

Oscillatory integrals in several variables

Oscillatory integrals of the 2^{nd} kind.
$\overline{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m, \ \overline{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$

$$\mathcal{I}(\overline{x}) = \int_{\mathbb{R}^n} e^{i\varphi(\overline{x},\overline{y})} \psi(\overline{x}, \overline{y})\, d\overline{y}$$

(the parameters $\overline{x}$ and the integration variables $\overline{y}$ are "intertwined" in the expressions for φ and ψ).

Examples. *Fourier transforms* $\hat{\psi}(\overline{x}) = \int_{\mathbb{R}^n} e^{-2\pi i \overline{x}\cdot\overline{y}} \psi(\overline{y})\, d\overline{y}$.

Fourier Integral Operator $T\psi(\overline{x}) = \int_{\mathbb{R}^n} e^{2\pi i \Phi(\overline{x},\overline{y})} a(\overline{x}, \overline{y})\, \hat{\psi}(\overline{y})\, d\overline{y}$ (sol. of PDEs)

AIM. Understand the nature of $\mathcal{I}(\overline{x})$ (depending on the nature of φ and ψ).

Tool needed.
Monomialize the phase while keeping track of the *different nature* of the variables $\overline{x}$ and $\overline{y}$.

Natural framework and natural tool:
Framework: φ, ψ subanalytic.
Tool: the Subanalytic Preparation Theorem.

Our framework: parametric integrals and subanalytic functions

Def. For $X \subseteq \mathbb{R}^m$ and $f : X \times \mathbb{R}^n \to \mathbb{R}$, define, $\forall \overline{x} \in X$ s.t. $f(\overline{x}, \cdot) \in L^1(\mathbb{R}^n)$,

the *parametric integral* $\mathcal{I}_f(\overline{x}) = \int_{\mathbb{R}^n} f(\overline{x}, \overline{y})\, d\overline{y}$.

Question. For $X \subseteq \mathbb{R}^m$ subanalytic and $f \in \mathcal{S}(X \times \mathbb{R}^n)$ s.t. $\forall \overline{x} \in X$ $f(\overline{x}, \cdot) \in L^1(\mathbb{R}^n)$, what is the nature of $\mathcal{I}_f$?

(Comte - Lion - Rolin). $f \in \mathcal{S}(X \times \mathbb{R}^n) \Rightarrow \mathcal{I}_f \in \mathcal{C}(X)$,
where $\mathcal{C}(X) :=$ $\mathbb{R}$-algebra generated by $\{g, \log h : \ g, h \in \mathcal{S}(X), \ h > 0\}$
("constructible" or "log-subanalytic" functions).

(Cluckers - D. Miller). $f \in \mathcal{C}(X \times \mathbb{R}^n) \Rightarrow \mathcal{I}_f \in \mathcal{C}(X)$.

AIM. Study *oscillatory integrals*

$$\mathcal{I}(\overline{x}) = \int_{\mathbb{R}^n} e^{i\varphi(\overline{x}, \overline{y})} \psi(\overline{x}, \overline{y})\, d\overline{y}, \text{ with } \varphi, \psi \in \mathcal{S}\left(\mathbb{R}^{m+n}\right)$$

Question. $\mathcal{D}(X) :=$ $\mathbb{C}$-algebra generated by $\mathcal{C}(X)$ and $\left\{e^{i\varphi(\overline{x})} : \ \varphi \in \mathcal{S}(X)\right\}$.

$$f \in \mathcal{D}(X \times \mathbb{R}^n) \overset{?}{\Rightarrow} \mathcal{I}_f \in \mathcal{D}(X)$$

The answer is **NO**: $\exists f \in \mathcal{D}(\mathbb{R} \times \mathbb{R})$ s.t. $\mathcal{I}_f \notin \mathcal{D}(\mathbb{R})$.

Example 1. Consider $f(x) = e^{-|x|}$ and its Fourier transform $\hat{f}(y)$.
A computation shows that $\hat{f}(y) = \frac{2}{1+4\pi^2 y^2} \in \mathcal{S}(\mathbb{R}) \cap L^1(\mathbb{R})$.
We can recover f by inverse Fourier transform of $\hat{f}$:
$f(x) = \int e^{2\pi i x y} \cdot \hat{f}(y)\, dy$, which is a parametric integral of a function in $\mathcal{D}(\mathbb{R})$.
Claim. $e^{-|x|} \notin \mathcal{D}(\mathbb{R})$. There are no *flat* functions in $\mathcal{D}(\mathbb{R})$.

Example 2. $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t} dt = \int_{\mathbb{R}} \frac{\chi_{[0,x]}(t)}{2it}\left(e^{it} - e^{-it}\right) dt$, which is a parametric integral of a function in $\mathcal{D}(\mathbb{R})$.
Claim. $\mathrm{Si}(x) \notin \mathcal{D}(\mathbb{R})$.
$\mathrm{Si}(x)$ has a *divergent* asymptotic expansion in the scale $\left\{\frac{\sin x}{x^{2k+1}}, \frac{\cos x}{x^{2k}} : \ k \in \mathbb{Z}\right\}$.

The key argument to prove the claims is the following
Remark. Let J be a finite set and $\forall j \in J$, let $c_j \neq 0$, $p_j(x)$ be distinct polynomials with $p_j(0) = 0$. Then $\sum_{j \in J} c_j e^{i p_j\left(x^{1/d}\right)} \not\to 0$ as $x \to +\infty$.
The remark can be proved using the theory of *almost periodic functions*.

One-dimensional transcendentals

Def. Consider the family of 1-dimensional integrals of the form:
$\gamma_{h,\ell}(\overline{x}) = \int_{\mathbb{R}} h(\overline{x},t)(\log|t|)^{\ell} e^{\mathrm{i}t} dt, \quad (\ell \in \mathbb{N},\ h \in \mathcal{S}(X \times \mathbb{R}),\ h(\overline{x},\cdot) \in L^1(\mathbb{R}))$
and $\mathcal{E}(X) :=$ the *$\mathcal{D}(X)$-module* generated by $\{\gamma_{h,\ell}\}_{h,\ell}$

MAIN THEOREM. $f \in \mathcal{E}(X \times \mathbb{R}^n) \Rightarrow \mathcal{I}_f \in \mathcal{E}(X)$. More precisely,
let $\mathrm{Int}(f,X) := \left\{\overline{x} \in X :\ f(\overline{x},\cdot) \in L^1(\mathbb{R}^n)\right\}$ (*integrability locus*).
Then there exists $F \in \mathcal{E}(X)$ s.t. $F(\overline{x}) = \displaystyle\int_{\mathbb{R}^n} f(\overline{x},\overline{y})\, d\overline{y} \quad \forall \overline{x} \in \mathrm{Int}(f,X)$
and there exists $g \in \mathcal{E}(X)$ s.t. $\mathrm{Int}(f,X) = \{\overline{x} \in X :\ g(\overline{x}) = 0\}$.

Corollary. $\mathcal{E}(X)$ is a $\mathbb{C}$-algebra.

Proof. By Fubini,
$\gamma_{h,\ell}(\overline{x}) \cdot \gamma_{h',\ell'}(\overline{x}) = \iint_{\mathbb{R}^2} h(\overline{x},t) \cdot h'(\overline{x},t') \cdot (\log|t|)^{\ell} \cdot (\log|t'|)^{\ell'} e^{\mathrm{i}(t+t')} dt dt'$,
which is the parametric integral of a function in $\mathcal{D}(X \times \mathbb{R}^2)$, and hence, by the Main Theorem, belongs to $\mathcal{E}(X)$. $\square$

Corollary. $\mathcal{E} = \bigcup \mathcal{E}(X)$ is the smallest collection of $\mathbb{C}$-algebras containing $\mathcal{S} \cup \left\{e^{\mathrm{i}\varphi} :\ \varphi \in \mathcal{S}\right\}$ and stable under parametric integration.

Generators of $\mathcal{E}$ and the proof of the Main Thm

Rem. An element of $\mathcal{E}(X \times \mathbb{R}^n)$ can be written as a finite sum of ***generators***:

$$T(\overline{x}, \overline{y}) = \psi(\overline{x}, \overline{y}) \cdot e^{\mathrm{i}\varphi(\overline{x}, \overline{y})} \cdot \gamma(\overline{x}, \overline{y}), \text{ where}$$

$$\psi \in \mathcal{C}(X \times \mathbb{R}^n), \ \varphi \in \mathcal{S}(X \times \mathbb{R}^n) \text{ and } \gamma(\overline{x}, \overline{y}) = \int_{\mathbb{R}} h(\overline{x}, \overline{y}, t)(\log|t|)^{\ell} e^{\mathrm{i}t} dt.$$

Proof of the Main Theorem.

• If $T \in \mathcal{D}(X \times \mathbb{R}^n)$ (i.e. $\gamma \equiv 1$) and T is integrable, then by o-minimality (cell decomposition, piecewise monotonicity, preparation) we can easily reduce to the case $\varphi(\overline{x}, \overline{y}) = y_1$ and show that $\int T \mathrm{d}\overline{y} \in \mathcal{E}(X)$.

• If $T \in \mathcal{E}(X \times \mathbb{R}^n)$ and $y \longmapsto |\psi(\overline{x}, \overline{y})| \int_{\mathbb{R}} \left| h(\overline{x}, \overline{y}, t)(\log|t|)^{\ell} \right| \mathrm{d}t \in L^1(\mathbb{R}^n)$, then by Fubini-Tonelli we can reduce to the previous step.

• **Core of the proof**: if $n = 1$ and $f = \sum T_j$ then we may suppose that each T_j is either as in the previous step or non-integrable. In the latter case, the γ_j in T_j *does not depend on* y ("naive" in y).
This uses the Subanalytic Preparation Theorem and other o-minimal tools.

• If each T_j is non-integrable and naive in y, then $\sum T_j$ is non-integrable.
This uses the theory of almost periodic functions.

Finite sums of exponentials of polynomials

Claim. Let J be a finite set and $\forall j \in J$ let $S_j(y) = c_j y^{r_j} (\log y)^{s_j} e^{ip_j\left(y^{\frac{1}{d}}\right)}$, where $c_j \in \mathbb{R}^*$, $r_j \in \mathbb{Q}$, d,$s_j \in \mathbb{N}$ and p_j are distinct polynomials with $p_j(0) = 0$.

Suppose that $\forall j \in J,\ S_j \notin L^1(\mathbb{R}^+)$. Then $\sum_{j\in J} S_j \notin L^1(\mathbb{R}^+)$.

Proof. Let $G(y) = \sum_{j\in J} c_j e^{ip_j\left(y^{\frac{1}{d}}\right)}$. Note that $y^{r_j}(\log y)^{s_j} > y^{-1}$ for $y \gg 0$.

Then, $\int_{\mathbb{R}^+} \left|\sum_{j\in J} S_j(y)\right| dy \geq \int_{\mathbb{R}^+} \frac{1}{y} |G(y)|\, dy$.

Since $G \not\equiv 0$, by continuity $\exists \varepsilon, \delta > 0$ s.t. $|G(y)| > \varepsilon$ on some interval I of length $\geq \delta$.

Idea: **If** G **were** ***periodic***, of period ν, then $|G| \geq \varepsilon$ on $V_\varepsilon := \bigcup_{k\in\mathbb{N}} (I + k\nu)$.

Then, $\int_{\mathbb{R}^+} \frac{1}{y} |G(y)|\, dy \geq \varepsilon \int_{\mathbb{R}^+ \cap V_\varepsilon} \frac{1}{y} dy \sim \sum_{k=1}^{\infty} \frac{\delta}{k\nu} = \infty.$

Now, G is not periodic. But, using the theory of *almost periodic functions* (H. Bohr), we show that the set $V_\varepsilon := \{y :\ |G(y)| \geq \varepsilon\}$ is **relatively dense** in $\mathbb{R}$, i.e. it intersects every interval of size ν (for some $\nu > 0$), and such an intersection has measure $\geq \delta$ (for some $\delta > 0$). $\square$

Almost periodic functions

Example. $f(x) = \sin(2\pi x) + \sin\left(2\sqrt{2}\pi x\right)$ is not periodic. However, $\forall \varepsilon > 0 \; \exists \; \infty$ many τ s.t. $x \in \mathbb{R} \; |f(x+\tau) - f(x)| < \varepsilon$.

Given f, an ε-period is a number τ such that $x \in \mathbb{R} \; |f(x+\tau) - f(x)| < \varepsilon$. $\mathcal{T}_{f,\varepsilon} := \{\tau : \; \tau \text{ is an } \varepsilon - \text{period}\}$.

Def. A continuous function f is **almost periodic** if for every $\varepsilon > 0$, the set $\mathcal{T}_{f,\varepsilon}$ is relatively dense, i.e. it intersects every interval of size ν (for some $\nu > 0$). This definition extends to $F : \mathbb{R}^n \to \mathbb{R}$.

Lemma. If $F : \mathbb{R}^n \to \mathbb{R}$ is almost periodic and $G(y) = F\left(y, y^2, \ldots, y^n\right)$, then $\exists \varepsilon > 0$ s.t. the set $V_\varepsilon := \{y : \; |G(y)| \geq \varepsilon\}$ intersects every interval of size ν (for some $\nu > 0$), and such an intersection has measure $\geq \delta$ (for some $\delta > 0$).

Recall: we have $G(y) = \sum_{j \in J} f_j e^{\mathrm{i} p_j(y)}$, which is not almost periodic, and we want to prove that $\displaystyle\int_{V_\varepsilon} \frac{1}{y} \mathrm{d}y = \infty$.

Apply the above lemma to $F(x) = \sum_{j \in J} f_j e^{\mathrm{i} L_j(x)}$, where $L_j(x_1, \ldots, x_n)$ is the linear form such that $p_j(y) = L_j\left(y, y^2, \ldots, y^n\right)$. $\square$