Topological complexity of symmetric semi-algebraic sets

Ordered Algebraic Structures and Related Topics

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A first slide

Notation

Throughout the talk let

- **R** be a real closed field (whose algebraic closure is **C**).
- For any finite set $\mathcal{P} \subset \mathbf{R}[X_1, \ldots, X_k]$ (respectively, $\mathcal{P} \subset \mathbf{C}[X_1, \ldots, X_k]$), we denote by $\mathrm{Zer}\left(\mathcal{P}, \mathbf{R}^k\right)$ (respectively $\mathrm{Zer}\left(\mathcal{P}, \mathbf{C}^k\right)$) the set of common zeros of $\mathcal{P}$ in $\mathbf{R}^k$(respectively $\mathbf{C}^k$).
- For a finite set $\mathcal{P} \subset \mathbf{R}[X_1, \ldots, X_k]$ a $\mathcal{P}$ -semi-algebraic set is a semi-algebraic subset of $\mathbf{R}^k$ defined by a quantifier-free formula with atoms of the form $P\{<, >, =\}0$ (resp. with $P \in \mathcal{P}$).

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Complexity of semi-algebraic sets

Uniform bounds on the number of connected components, Betti numbers etc. in terms of:

- The number of polynomials (combinatorial complexity)
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Dramatis personae

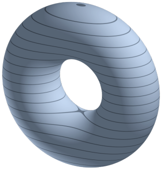
For a semi-algebraic set $S \subset \mathbf{R}^k$, and any field of coefficients $\mathbb{F}$, we denote for $i \geqslant 0$:

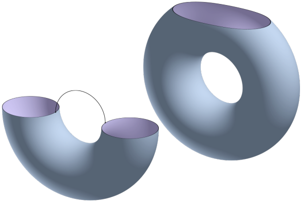
- $b_i(S, \mathbb{F}) = \dim_{\mathbb{F}} H_i(S, \mathbb{F})$,
- $b(S, F) = \sum_{i \geqslant 0} b_i(S, \mathbb{F})$,

where $H_i(S, \mathbb{F})$ is the i-th homology group of S with coefficients in $\mathbb{F}$.

Betti numbers

- $b_0(S)$ = the number of connected components.
- $i \geqslant 1$, $b_i(S)$ the number of i-cycles that do not bound.





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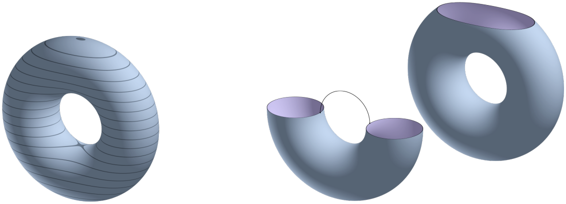

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Bounds for Betti numbers

Theorem (Petrovskiĭ, Oleĭnik, Thom, Milnor)

$\sum_i b_i(\mathrm{Zer}(\mathcal{P}, \mathbf{R}^k), \mathbb{F}) \leqslant d(2d-1)^{k-1} = (O(d))^k$

By taking real and imaginary parts one gets

$$\sum_i b_i(\mathrm{Zer}(\mathcal{P}, \mathbf{C}^k), \mathbb{F}) \leqslant d(2d-1)^{2k-1} = (O(d))^{2k}$$

Theorem (Basu, Pollack, Roy)

Let $s := |\mathcal{P}|$, $d := \max_{p \in \mathcal{P}} \deg p$ and $S \subset \mathbf{R}^k$ be a $\mathcal{P}$-semi-algebraic set. Then,

$$\sum_i b_i(S, \mathbb{F}) = \sum_{i=0}^{k} \sum_{j=1}^{k-i} \binom{s}{j} 6^j d(2d-1)^{k-1} = (O(sd))^k.$$

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- The bounds are exponential in k, for fixed d.
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A Meta-Belief

Belief

The worst case topological complexity of a class of semi-algebraic sets (measured by the Betti numbers for example) can serve as a rough lower bound for the complexity of algorithms for computing topological invariants or deciding topological properties of this class of sets.

Example

- *It is NP-hard to decide to decide if a real algebraic variety defined by one polynomial of degree 4 is empty or not - and correspondingly the Betti-numbers of such sets can be exponential.*
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Notational setup

The symmetry

- Let S_k denote the symmetric group.
- It acts on $\mathbf{R}^k$ by permuting coordinates.
- A polynomial F with $F(X) = F(\sigma(X)) \forall \sigma \in S_{\mathbf{k}}$ is called symmetric.

More general setup:

- For $m \in \mathbb{N}$ take $\mathbf{R}^{m \cdot k}$. Then S_k operates by permuting m-tuples of coordintes.
- Let $\mathbf{k} = (k_1, \ldots, k_\omega) \in \mathbf{Z}^{\omega}_{>0}$, $k = \sum_{i=1}^{\omega} k_i$ and look at the product $S_{\mathbf{k}} = S_{k_1} \times \cdots \times S_{k_\omega}$ and each S_{k_i} operates on blocks of variables

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Topological complexity

A counter example

Let F symmetric of degree d and consider $V_{\mathbf{R}} = \mathrm{Zer}\left(\{P\}, R^k\right)$.
Then, one can check $V_{\mathbf{R}} = \emptyset$ in time polynomial in k.

However:

- Look at

$$P = \sum_{i=1}^{k} \left(\prod_{j=1}^{d} (X_i - j) \right)^2 .$$

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Question

How to reconcile this example with the "Meta-belief"?

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Equivariant Betti numbers

- Let $X_{\mathbf{R}} \subset \mathbf{R}^k$ be a semi algebraic set and $X_{\mathbf{C}}$ be an algebraic subset of $\mathbf{C}^k$, such that S_k operates on $X_{\mathbf{R}}$ resp. $X_{\mathbf{C}}$ (by permuting coordinates).
- Denote by $X_{\mathbf{C}}/S_k$ / $X_{\mathbf{R}}/S_k$ the *orbit space* of this action.
- If $\mathrm{char}(\mathbb{F}) = 0$ then $H^*(X/S_k, \mathbb{F})$ is isomorphic to the so called equivariant cohomology $H^*_{S_k}(X, \mathbb{F})$ (Borel construction).
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- Let us look again at the example

$$P = \sum_{i=1}^{k} \left(\prod_{j=1}^{d} (X_i - j) \right)^2 .$$

- We find $b_0 \left(V_{\mathbb{R}} / S_k, \mathbb{Q} \right) = \sum_{\ell=1}^{d} p(k, \ell) \leqslant O(k^d)$, where $p(k, \ell)$ denotes the number of partitions of n with exactly ℓ parts.

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1. $$b\left(V_{\mathbf{C}}/S_k, \mathbb{F}\right) \leqslant d^{O(d)}$$

2. $$b(V_{\mathbf{R}}/S_k, \mathbb{Q}) \leqslant O(k^{2d-1}).$$

3. *In addition we have for all* $i \geqslant \min(2d, k)$

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Further let $S \subset \mathbf{R}^k$ be a semi algebraic set defined with polynomials in $\mathcal{P}$ and let $S/\mathfrak{S}_k$ denote the quotient space. Then

$$b(S/\mathfrak{S}_k, \mathbb{Q}) \leqslant O(s^{5d} k^{4d-1}).$$

In addition, $b_i(V_{\mathbf{R}}/\mathfrak{S}_k, \mathbb{Q}) = 0$ and $b_i(S/\mathfrak{S}_k, \mathbb{Q}) = 0$ for all $i \geqslant 5d$.

Ideas behind the proof

The proof of the statements relies on two ingredients:

1. Equivariant Morse-Theory
2. Understanding of the stabilizers of critical points of a particular symmetric Morse function

As a consequence of our methods we obtain new algorithms for computing the generalized Euler-Poincaré characteristic of semi-algebraic sets defined in terms of symmetric polynomials.These algorithms have complexity which is polynomial (for fixed degrees and the number of blocks) in the number of symmetric variables.

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An application

Let $\mathcal{P} \in \mathbf{R}[Y_1, \ldots, Y_m, X_1, \ldots, X_k]$ with $\deg(P) \leqslant d$ and let S be a $\mathcal{P}$ semi algebraic set, which is closed and bound. Denote by $\pi : \mathbf{R}^{m+k} \longrightarrow \mathbf{R}^m$ be the projection map to the first m co-ordinates.

Theorem (Gabrielov, Vorobjov, Zell'08)

With the above notation

$$b(\pi(S), \mathbb{Q}) \leqslant \sum_{0 \leqslant p < m} b(\underbrace{S \times_\pi \cdots \times_\pi S}_{p+1}, \mathbb{Q}),$$

where $\underbrace{S \times_\pi \cdots \times_\pi S}_{p+1}$ denotes the p-fold fibered product of S.

Corollary

$$b(V, \mathbb{Q}) \leqslant O(d)^{(k+1)m}.$$

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where $\underbrace{S \times_\pi \cdots \times_\pi S}_{p+1}$ *denotes the p-fold fibered product of* S.

Corollary

$$b(V, \mathbb{Q}) \quad \leqslant \quad O(d)^{(k+1)m}.$$

An application

Let $\underbrace{X \times_f \cdots \times_f X}_{p+1}$ denote the p fold fiber product of X with f. There is a natural action of S_{p+1} which permutes the factors. For each $p > 0$ we denote by $\mathrm{Sym}_f^{(p)}(X), \mathbb{F})$ the associated quotient $\underbrace{X \times_f \cdots \times_f X}_{p+1} / S_{p+1}$.

Theorem

For any field of coefficients $\mathbb{F}$, there exists a spectral sequence converging to $H_*(Y, \mathbb{F})$ whose E_1-term is given by

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An application

Theorem

Let $P \in \mathbf{R}[Y_1, \ldots, Y_n, X_1, \ldots, X_m]$ *be a non-negative polynomial and with* $\deg(P) \leqslant d$. *Let* $V := \{x \in \mathbf{R}^{n+m} \; : \; P(x) = 0\}$ *be bounded, and*

$$\pi : \mathbf{R}^m \times \mathbf{R}^k \longrightarrow \mathbf{R}^m$$

be the projection map to the first m coordinates. Then,

$$b(\pi(V), \mathbb{F}) \leqslant m^{(2d)^k} (O(d))^{m+k(2d)^k+1}.$$

Group-Representation

- Let G be a group. Then a homomorphism $\phi : G \to \mathrm{GL}(V)$ for some $\mathbb{F}$ vector space V. is called a *representation of G*. Equivalently, V is said to be a G-module.
- If V contains only trivial G modules, V is called *irreducible*.
- We denote the equivalence classes of irreducible modules of G by $\mathrm{Irred}(G, \mathbb{F})$.
- Every G-module can be decomposed into irreducibles, i.e.,

$$V \simeq \bigoplus_{W_i \in \mathrm{Irred}(G,\mathbb{F})} m_i W_i,$$

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Action on a space

Let X be a topological space and G be a finite group acting on X.

- The action of G on X induces an action of G on $\mathrm{H}^*(X, \mathbb{F})$, which turns $\mathrm{H}(X, \mathbb{F})$ into a G-module.
- If $\mathrm{char}(\mathbb{F}) = 0$ then

$$\mathrm{H}^*(X/G, \mathbb{F}) \xrightarrow{\sim} \mathrm{H}^*_G(X, \mathbb{F}) \xrightarrow{\sim} (\mathrm{H}^*(X, \mathbb{F}))^G.$$

- So the bound on the S_k-equivariant Betti numbers is in fact a bound on the multiplicity of the *trivial* representation of S_k.

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Specht-Modules

- The irreducible representations of S_k are $1:1$ with the partitions of k and denoted by $\mathfrak{S}^\lambda$.
- Let $(\lambda_1, \ldots, \lambda_l) \vdash k$ then so called *Young-module* is

$$M^\lambda := \mathrm{Ind}_{S_{\lambda_1} \times \cdots \times S_{\lambda_l}}^{S_k} \mathbf{1}.$$

- For each $\lambda \vdash k$ Young's rule gives

$$M^\lambda = \bigoplus_{\mu \vdash k} K(\lambda, \mu) \mathfrak{S}^\mu,$$

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Example

- Let $F_k := \sum_{i=1}^{k}(X_i^2(X_i-1)^2 - \varepsilon$ and consider

$$V_k := \mathrm{Zer}(F_k, \mathbf{R}^k).$$

- Then

$$\mathrm{H}^0(V_k,\mathbb{F}) \simeq \bigoplus_{0 \leqslant i \leqslant k} \mathrm{H}^0(V_{k,i},\mathbb{F}),$$

where for $0 \leqslant i \leqslant k$ $V_{k,i}$ is the S_k-orbit of the connected component of V_k which is infinitesimally close to the point

$$x^i := (\underbrace{0,\ldots,0}_{i},\underbrace{1,\ldots,1}_{k-i}),$$

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Example

Observing that $\mathrm{H}^0(V_{k,i},\mathbb{F}) \simeq M^\lambda$, where $\lambda = (n-i,i)$ (or $(i,n-i)$) we get

$$\mathrm{H}^0(V_k,\mathbb{F}) \simeq \bigoplus_{\mu \vdash k} m_{0,\mu}\mathfrak{S}^\mu,$$

with $m_{0,\mu} = 0$ for all μ with $\ell(\mu) > 2$ and $m_{0,\mu} = \mu_1 - \mu_2 + 1 \leqslant k$ for all μ with $\ell(\mu) \leqslant 2$.

Example

Equivariant Poincaré duality

Theorem

Let $V \subset \mathbf{R}^k$ be a bounded smooth compact semi-algebraic oriented hyper surface which is stable under the action of S_k on $\mathbf{R}^k$. Then, for each $p, 0 \leqslant p \leqslant k$ there is a S_k-isomorphism

$$\mathrm{H}^p(V, \mathbb{F}) \xrightarrow{\sim} \mathrm{H}^{k-p-1}(V, \mathbb{F}) \otimes \mathrm{sign}_k .$$

Let $\mu' \vdash k$ denote the transpose of $\mu \vdash k$ then this implies in our example that

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Complexity of the Isotypic-decomposition

Theorem

Let $P \in \mathbf{R}[X_1, \ldots, X_k]$ *be a symmetric polynomial with* $\deg(P) = d$. *Let* $V = \mathrm{Zer}(P, \mathbf{R}^k)$. *Consider the decomposition*

$$\mathrm{H}^*(V, \mathbb{Q}) = \bigoplus_{\mu \vdash k} m_\mu \mathfrak{S}^\mu.$$

Then:

1. $m_\mu \neq 0$ *implies that* μ *has at most* 2d *'long rows" and* 2d *"long columns".*
2. *The number of such partitions is bounded by a polynomial in* k.
3. *Further,* $m_\mu \leqslant k^{O(d^2)} d^d$.

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A conjecture on representational stability

Let $f \in \mathbf{R}[X_1, \ldots, X_d]$ be a symmetric polynomial of degree d. Define

$$F_k = \phi_{d,k}(F) \in \mathbf{R}[X_1, \ldots, X_k]^{\mathfrak{S}_k},$$

with the canonical injection

$$\phi_{d,k} : \mathbf{R}[X_1, \ldots, X_d]^{\mathfrak{S}_d}_{\leqslant d} \hookrightarrow \mathbf{R}[X_1, \ldots, X_k]^{\mathfrak{S}_k},$$

and consider $V_k := \mathrm{Zer}(F_k, \mathbf{R}^k)$ and a resulting sequence of homology groups $(\mathrm{H}^*(V_k, \mathbb{Q}))_n$. Fix $k_0 \in \mathbb{N}$, $\mu = (\mu_1, \ldots, \mu_\ell) \vdash k_0$ and define for $k \geqslant k + \mu_1$

$$\{\mu\}_k = (k - k_0, \mu_1, \mu_2, \ldots, \mu_\ell) \vdash k. \quad (1)$$

Let $p \geqslant 0$. Does there exists a polynomial $P_{F,p,\mu}(k)$ such that $m_{p,\mu_k}(V_k, \mathbb{F}) = P_{F,p,\mu}(k)$?!

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Algorithmic consequences

- Aim: Design polynomial-time algorithms which compute the m_λ!

Fine

Merci, thanks and
paljon kiitoksia!

ArXiv

1. Bounding the equivariant Betti numbers and computing the generalized Euler-Poincaré characteristic of symmetric semi-algebraic sets. (with S. Basu): *arXiv:1312.6582.*
2. On the isotypic decomposition of homology modules of symmetric semi-algebraic sets. (with S. Basu) :*arXiv:1503.00138.*