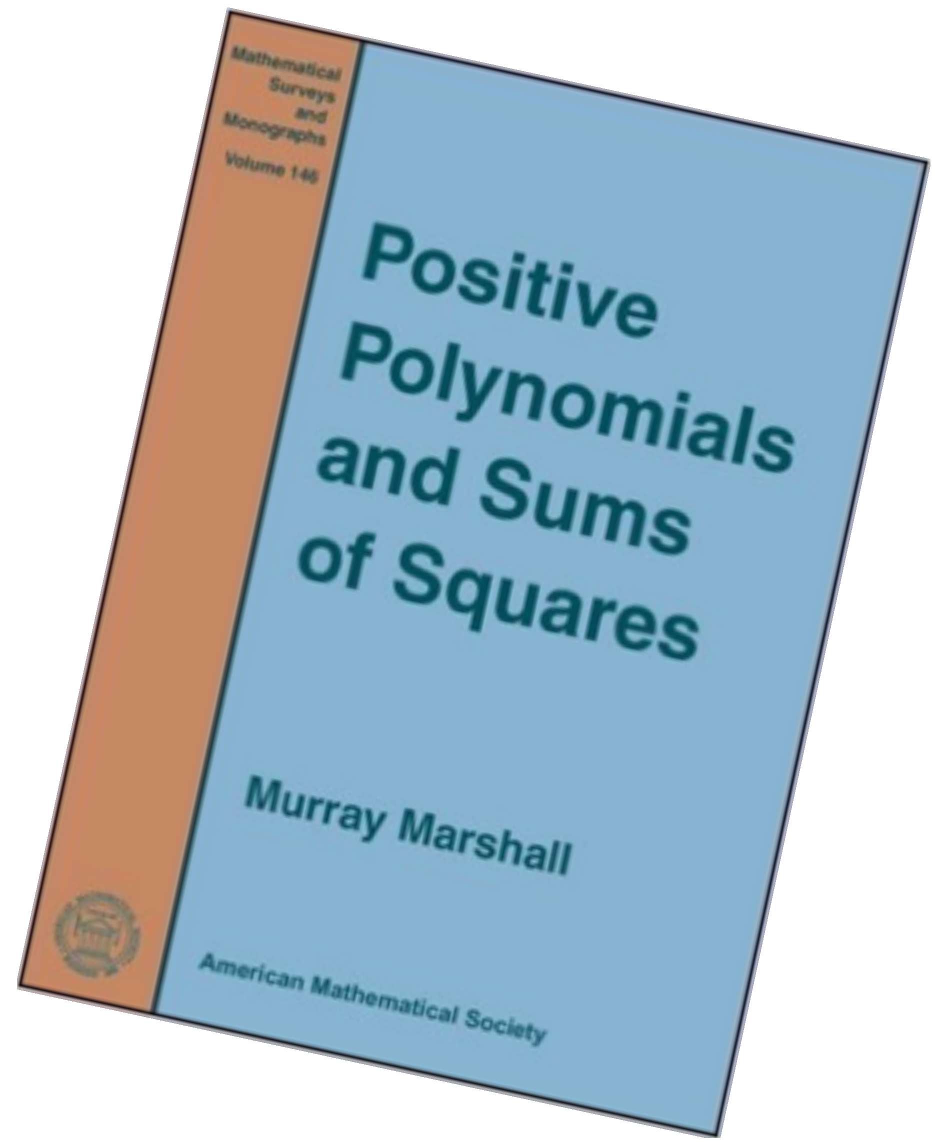
Positive Polynomials
according to Murray Marshall



Daniel Plaumann
Universität Konstanz

Ordered Algebraic Structures and Related Topics
Luminy, 16 October 2015

Murray Marshall's work on positive polynomials

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Murray's magic to finish the proof

$$f(x,y) \geq \epsilon(x)(1+y^2)^d$$

$$f_1(x,y) := f(x,y) - \epsilon(x)(1+y^2)^d$$

$$f_1(x,y) =$$

$$\sum_{k=1}^{\ell}(\sum_{j=1}^{2} h_{0jk}(x,y)^2 + \sum_{j=1}^{2} h_{1jk}(x,y)^2 x + \sum_{j=1}^{2} h_{2jk}(x,y)^2(1-x)) + \sum_{i=0}^{2d} b_i(x)y^i,$$

$b_i(x) \in \mathbb{R}[x]$, $|b_i(x)| \leq \frac{2}{5}\epsilon(x)$ on $[0,1]$, $i = 0, \cdots, 2d$. Combining this with $f(x,y) = f_1(x,y) + \epsilon(x)(1+y^2)^d$ yields $f(x,y) = s_1(x,y) + s_2(x,y) + s_3(x,y)$, where

$$s_1(x,y) := \sum_{k=1}^{\ell}(\sum_{j=1}^{2} h_{0jk}(x,y)^2 + \sum_{j=1}^{2} h_{1jk}(x,y)^2 x + \sum_{j=1}^{2} h_{2jk}(x,y)^2(1-x)),$$

$$s_2(x,y) := \frac{2}{5}\epsilon(x)(2 + y + 3y^2 + y^3 + 3y^4 + \cdots + y^{2d-1} + 2y^{2d}) + \sum_{i=0}^{2d} b_i(x)y^i,$$

$$s_3(x,y) := \epsilon(x)[(1+y^2)^d - \frac{2}{5}(2 + y + 3y^2 + y^3 + 3y^4 + \cdots + y^{2d-1} + 2y^{2d})].$$

Murray's magic to finish the proof

Let T denote the preordering of $\mathbb{R}[x,y]$ generated by $x(1-x)$. As pointed out earlier, $x, 1-x \in T$. Clearly $s_1(x,y) \in T$. The argument in [3, Th. 5.1] shows that $s_2(x,y) \in T$. In more detail, since $|b_i(x)| \leq \frac{2}{5}\epsilon(x)$ on $[0,1]$, $\frac{2}{5}\epsilon(x) \pm b_i(x) \in T$, by [3, Th. 2.2] or [4, Prop. 2.7.3], for $i = 0, \cdots, 2d$. This yields

$$\frac{2}{5}\epsilon(x)y^i + b_i(x)y^i \in T, \text{ for } i \text{ even.} \tag{5.1}$$

For i odd, say $i = 2m+1$, use the identity $y^{2m+1} = \frac{1}{2}y^{2m}((y+1)^2 - y^2 - 1)$ plus the fact that $\frac{2}{5}\epsilon(x)y^{2m}(y+1)^2 + b_i(x)y^{2m}(y+1)^2$, $\frac{2}{5}\epsilon(x)y^{2m}y^2 - b_i(x)y^{2m}y^2$ and $\frac{2}{5}\epsilon(x)y^{2m} - b_i(x)y^{2m}$ all belong to T to obtain

$$\frac{2}{5}\epsilon(x)(y^{i+1} + y^i + y^{i-1}) + b_i(x)y^i \in T, \text{ for } i \text{ odd.} \tag{5.2}$$

Adding together the various terms of type (5.1) and (5.2), for $i = 0, \cdots, 2d$, we see that $s_2(x,y) \in T$.

Murray's magic to finish the proof

The fact that $s_3(x, y)$ belongs to T follows from the identity

$$
\begin{aligned}
&(1+y^2)^d - \frac{2}{5}(2 + y + 3y^2 + y^3 + 3y^4 + \cdots + y^{2d-1} + 2y^{2d}) \\
&= (1+y^2)^d + \frac{1}{5}(1 + y^2 + \cdots + y^{2d-2})(1-y)^2 \\
&- \frac{8}{5}(y^2 + y^4 + \cdots + y^{2d-2}) - (1 + y^{2d}) \\
&= \frac{1}{5}(1 + y^2 + \cdots + y^{2d-2})(1-y)^2 + \sum_{i=1}^{d-1}(\binom{d}{i} - \frac{8}{5})y^{2i}.
\end{aligned}
$$

This means, finally, that $f(x, y) = s_1(x, y) + s_2(x, y) + s_3(x, y) \in T$. $\square$

Summer School in Tame Geometry

University of Konstanz, July 18-23 2016

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Organisers: Pantelis Eleftheriou (Konstanz), Salma Kuhlmann (Konstanz), Daniel Plaumann (Konstanz), Jonathan Pila (Oxford), Margaret Thomas (Konstanz)