K ordered field, R real closed field, $K \subseteq R$.

$\mathcal{H}$ system of sign conditions in $K[x_1,.., x_k]$

$$\mathcal{H} = \begin{cases} P_1 \neq 0 \\ \vdots \\ P_{m_1} \neq 0 \\ Q_1 \geq 0 \\ \vdots \\ Q_{m_2} \geq 0 \\ R_1 = 0 \\ \vdots \\ R_{m_3} = 0 \end{cases}$$

$$\left(\begin{array}{lll} S \leq 0 & \longrightarrow & -S \geq 0 \\ S > 0 & \longrightarrow & \begin{cases} S \neq 0 \\ S \geq 0 \end{cases} \\ S < 0 & \longrightarrow & \begin{cases} S \neq 0 \\ -S \geq 0 \end{cases} \end{array} \right)$$

Positivstellensatz (Krivine '64 - Stengle '74)

$\mathcal{H}$ no solutions in $R^k \Leftrightarrow \exists\ S, N, Z \in K[x_1,.., x_k]$ with

$$S = \prod_{i=1}^{m_1} P_i^{2e_i}, \quad N = \sum_{I \subseteq \{1,..,m_2\}} \overset{\in K_{>0}}{\omega_i} \Big(\sum_j R_{I,j}^2 \Big) \prod_{i \in I} Q_i, \quad Z = \sum_{i=1}^{m_3} H_i R_i$$

such that $\underbrace{S}_{>0} + \underbrace{N}_{\geq 0} + \underbrace{Z}_{=0} = 0 \quad \leftarrow$ incompatibility of $\mathcal{H}$.

$\downarrow \mathcal{H} \downarrow$.

degree of $\downarrow \mathcal{H} \downarrow$: max degree of S, $R_{I,j}^2 \prod_{i \in I} Q_i$, $H_i R_i$

Example:

$$\mathcal{H} \begin{cases} x_1 \neq 0 \\ x_2 - x_1^2 - 1 \geq 0 \\ x_1 x_2 = 0 \end{cases}$$

$$\overbrace{x_1^2}^{S} + \overbrace{x_1^2(x_2 - x_1^2 - 1) + x_1^4}^{N} + \overbrace{(-x_1^2 x_2)}^{Z} = 0$$

$\downarrow \mathcal{H} \downarrow$ degree 4.

Positivstellensatz implies Real Nullstellensatz:

$p, p_1, \dots, p_s \in \mathbb{R}[x_1, \dots, x_k]$, p vanishes on $\{x \in \mathbb{R}^k / p_1 = \dots = p_s = 0\}$.

$$\Rightarrow \begin{cases} p \neq 0 \\ p_1 = 0 \\ \vdots \\ p_s = 0 \end{cases} \text{ no solution in } \mathbb{R}^k \underset{\text{Positivstellensatz}}{\Rightarrow} p^{2e} + \underset{\text{sum of squares}}{N} + \sum_{i=1}^{s} M_i p_i = 0 \Rightarrow$$

$$\Rightarrow p \in \sqrt[R]{(p_1, \dots, p_s)}$$

Positivstellensatz implies Hilbert 17th problem:

$p \in \mathbb{R}[x_1, \dots, x_k]$, $p \geq 0$ in $\mathbb{R}^k \Rightarrow \{ p < 0$ no solution in $\mathbb{R}^k \Rightarrow$

$$\Rightarrow \begin{cases} p \neq 0 \\ -p \geq 0 \end{cases} \text{ no solution in } \mathbb{R}^k \underset{\text{Positivstellensatz}}{\Rightarrow} p^{2e} + \left(\sum R_i^2\right) + \left(\sum P_i^2\right)(-p) = 0 \Rightarrow$$

$$\Rightarrow p = \frac{\left(\sum P_i^2\right) p^2}{p^{2e} + \left(\sum R_i^2\right)} = \frac{\left(\sum P_i^2\right) p^2 \left(p^{2e} + \sum R_i^2\right)}{\left(p^{2e} + \left(\sum R_i^2\right)\right)^2} \leftarrow \text{sum of squares of rational functions}$$

Original proofs of Positivstellensatz: * Non-constructive

* No degree bound.

Lombardi '90: * Constructive proof

* degree bound

$$\left.2^{2^{\cdot^{\cdot^{2^{d\log(d)+\log(\log(s))+c}}}}}\right\}k+4$$

primitive recursive degree bound

$d=$ max deg $P_1,\dots P_{m_1}, f_1,\dots f_{m_2}, h_1,\dots h_{m_3}$,

$s=\#\{P_1,\dots P_{m_1}, f_1,\dots f_{m_2}, h_1,\dots h_{m_3}\}$

$k=\#$ variables

$c=$ universal constant.

Lombardi, P., Roy: * constructive proof.

* degree bound for Positivstellensatz

$$2^{2^{\left(2^{\max\{2,d\}^{4^k}}+s^{2^k}\max\{2,d\}^{16^k\mathrm{bit}(d)}\right)}}$$

* degree bound for Hilbert 17th problem

$$2^{2^{2^{d^{4^k}}}}$$

elementary recursive degree bounds

Weak inferences

First example: $A \neq 0 \Rightarrow A<0 \vee A>0$.

If systems, $\downarrow \mathcal{H}, A<0 \downarrow \;\Rightarrow\; \begin{cases} \mathcal{H}(x) \\ A(x)<0 \end{cases}$ no solution in $\mathbb{R}^k$

$\downarrow \mathcal{H}, A>0 \downarrow \;\Rightarrow\; \begin{cases} \mathcal{H}(x) \\ A(x)>0 \end{cases}$ no solution in $\mathbb{R}^k$

$\Downarrow$

$\downarrow \mathcal{H}, A(x)\neq 0 \downarrow \;\Leftarrow\; \begin{cases} \mathcal{H}(x) \\ A(x)\neq 0 \end{cases}$ no solution in $\mathbb{R}^k$

we want a direct way to obtain from the incompatibilities above, the one below.

If we know how to do this, we say we have a weak inference:

$$A \neq 0 \;\vdash\; A<0 \vee A>0$$

$$\downarrow \mathcal{H}, A<0 \downarrow \;\to\; \downarrow \mathcal{H}, A\neq 0, -A\geq 0 \downarrow \;\to\; S_1A^{2e_1}+N_1+Z_1 = AM_1$$

$$\downarrow \mathcal{H}, A>0 \downarrow \;\to\; \downarrow \mathcal{H}, A\neq 0, A\geq 0 \downarrow \;\to\; S_2A^{2e_2}+N_2+Z_2 = -AM_2$$

$$\underbrace{S_1S_2A^{2e_1+2e_2}}_{S} + \underbrace{\left(N_1\left(S_2A^{2e_2}+N_2\right) + S_1A^{2e_1}N_2 + A^2M_1M_2\right)}_{N} + \underbrace{\left(Z_1\left(S_2A^{2e_2}+N_2+Z_2\right) + Z_2\left(S_1A^{2e_1}+N_1\right)\right)}_{Z} = 0$$

More interesting weak inference

+) $P \in K[x_1, .. x_k, y]$, monic in y, odd degree in y

$$\vdash \exists y \;/\; P(x, y) = 0.$$

+) $P \in K[x_1, .. x_k, y]$, monic in y

$$\vdash \exists z = a + ib \;/\; \underbrace{P(x, z) = 0}_{P_{Re}(x, a, b) = P_{Im}(x, a, b) = 0}.$$

In general: given $\mathcal{P} \subseteq K[X_1,.., X_k]$

$$\mathcal{P}_k = \mathcal{P} \subseteq K[X_1,.., X_k]$$

$$\mathcal{P}_{k-1} = \mathrm{Elim}(\mathcal{P}_k) \subseteq K[X_1,.., X_{k-1}]$$

$$\mathcal{P}_1 = \mathrm{Elim}(\mathcal{P}_2) \subseteq K[X_1]$$

$$\mathcal{P}_0 = \mathrm{Elim}(\mathcal{P}_1) \subseteq K.$$

We produce for every realizable sign condition τ on $\mathcal{P}_i$,

$$\mathrm{sign}(\mathcal{P}_i) = \tau \vdash \bigvee \mathrm{sign}(\mathcal{P}_{i-1}) = \sigma.$$

σ realizable sign condition on $\mathcal{P}_{i-1}$

given that $\mathrm{sign}(\mathcal{P}_i) = \tau$

Finally, given a system $\mathcal{H}$ with no solutions in R^k, we take $\mathcal{P} = \{P_1,.., P_{m_1}, R_1,.., R_{m_2}, R_1,.., R_{m_3}\}$ and for every realizable sign condition σ on $\mathcal{P}_k = \mathcal{P}$, we produce easily an incompatibility

$$\downarrow \mathrm{sign}(\mathcal{P}_k) = \sigma, \mathcal{H} \downarrow.$$

After "eliminating" all variables, we obtain for the only realizable sign condition σ on $\mathcal{P}_0$, an incompatibility

$$\downarrow \mathrm{sign}(\mathcal{P}_0) = \sigma, \mathcal{H} \downarrow \longrightarrow \downarrow \mathcal{H} \downarrow$$