

K ordered field, R real closed field, $K \subseteq R$.

(1)

H system of sign conditions in $K[x_1, \dots, x_n]$

$$H = \left(\begin{array}{l} P_1 \neq 0 \\ \vdots \\ P_{m_1} \neq 0 \\ \Phi_1 \geq 0 \\ \vdots \\ \Phi_{m_2} \geq 0 \\ R_1 = 0 \\ \vdots \\ R_{m_3} = 0 \end{array} \right) \left(\begin{array}{l} S \leq 0 \rightarrow -S \geq 0 \\ S \geq 0 \rightarrow \left. \begin{array}{l} S \neq 0 \\ S \geq 0 \end{array} \right\} \\ S \leq 0 \rightarrow \left. \begin{array}{l} S \neq 0 \\ -S \geq 0 \end{array} \right\} \end{array} \right)$$

Positivstellensatz (Krivonozhko '64 - Stengle '74)

H no solutions in $R^k \Leftrightarrow \exists S, N, Z \in K[x_1, \dots, x_n]$ with

$$S = \prod_{i=1}^{w_1} P_i^{2e_i}, \quad N = \sum_{I \in \{1, \dots, w_2\}} w_i \left(\sum_{j \in I} R_{I,j}^2 \right) \prod_{i \in I} \Phi_i, \quad Z = \sum_{i=1}^{w_3} H_i R_i$$

seek that $\underbrace{S}_{\geq 0} + \underbrace{N}_{\geq 0} + \underbrace{Z}_{=0} = 0 \Leftrightarrow$ incompatibility of H .

$\downarrow H \downarrow$

degree of $\downarrow H \downarrow$: max degree of $S, \sum_{i \in I} R_{I,j}^2 \prod_{i \in I} \Phi_i, H_i R_i$.

Example: $H = \left(\begin{array}{l} x_1 \neq 0 \\ x_2 - x_1^2 - 1 \geq 0 \\ x_3 x_2 = 0 \end{array} \right)$

$$\underbrace{\frac{S}{x_1^2}} + \underbrace{\frac{N}{x_1^2(x_2 - x_1^2 - 1) + x_3^4}} + \underbrace{\frac{Z}{(-x_1^2 x_2)}} = 0$$

$\downarrow H \downarrow$ degree 4.

Positivstellensatz implies Real Nullstellensatz:

(2)

$P, P_1, \dots, P_s \in \mathbb{R}[x_1, \dots, x_n]$, P non-zero on $\{x \in \mathbb{R}^n / P_1 = \dots = P_s = 0\}$.

$$\Rightarrow \begin{cases} P \neq 0 \\ P_1 = 0 \\ \vdots \\ P_s = 0 \end{cases} \text{ no solution in } \mathbb{R}^n \Rightarrow P^{2c} + N + \sum_{i=1}^s \lambda_i P_i = 0 \Rightarrow$$

\downarrow positivstellensatz \uparrow sum of squares

$$\Rightarrow P \in \sqrt{\mathbb{R}[P_1, \dots, P_s]} \quad \checkmark$$

Positivstellensatz implies Hilbert 17th problem:

$P \in \mathbb{R}[x_1, \dots, x_n]$, $P \geq 0$ in $\mathbb{R}^n \Rightarrow \begin{cases} P \geq 0 \\ \text{no solution in } \mathbb{R}^n \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} P \geq 0 \\ -P \geq 0 \end{cases} \text{ no solution in } \mathbb{R}^n \Rightarrow P^{2c} + (\sum R_i^2) + (\sum Q_i^2)(-P) = 0 \Rightarrow$$

\downarrow positivstellensatz

$$\Rightarrow P = \frac{(\sum R_i^2) P^2}{P^{2c} + (\sum R_i^2)} = \frac{(\sum R_i^2) P^2 (P^{2c} + \sum R_i^2)}{(P^{2c} + (\sum R_i^2))^2} \quad \text{sum of squares of rational functions}$$

Original proofs of Positivstellensatz: + Non-constructive
+ No degree bound.

Low bound '90: + Constructive proof
+ degree bound

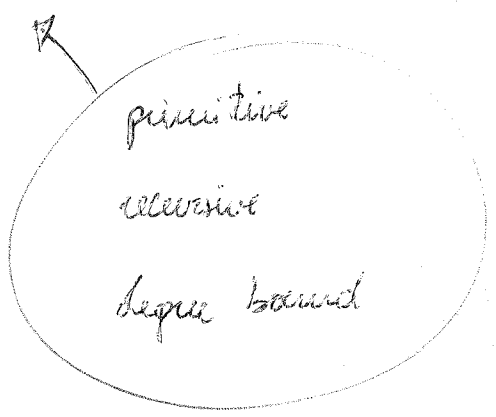
$$2^{2^{d \log(d) + \log(\log(s)) + c}} \left. \vphantom{2^{2^{d \log(d) + \log(\log(s)) + c}}} \right\} k+4$$

$d = \max \deg \{p_1, \dots, p_{m_1}, f_1, \dots, f_{m_2}, r_1, \dots, r_{m_3}\}$

$S = \# \{p_1, \dots, p_{m_1}, f_1, \dots, f_{m_2}, r_1, \dots, r_{m_3}\}$

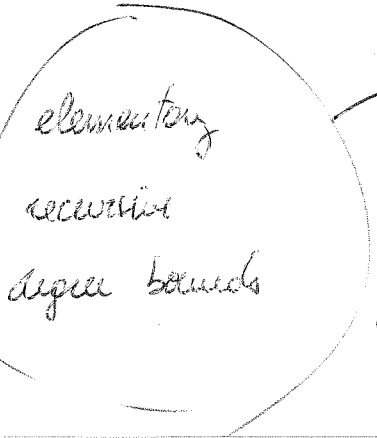
$k = \# \text{ variables}$

$c = \text{universal constant}$



double bound, P., Nag: + constructive proof.

+ degree bound for Positivstellensatz



$$2^{2^2 \left(\max\{2, d\}^{16^k} + 5^{2^k} \max\{2, d\}^{16^k} \log(d) \right)}$$

+ degree bound for Kalbarczyk's problem

$$2^{2^{2^{d^k}}}$$

Weak inferences

First example: $A \neq 0 \Rightarrow A < 0 \vee A > 0$

If system $\downarrow H, A < 0 \downarrow \Rightarrow \begin{cases} H(x) \\ A(x) < 0 \end{cases}$ no solution in \mathbb{R}^k

$\downarrow H, A > 0 \downarrow \Rightarrow \begin{cases} H(x) \\ A(x) > 0 \end{cases}$ no solution in \mathbb{R}^k

we want a direct way to obtain from

$\downarrow H, A(x) \neq 0 \downarrow \Leftrightarrow \begin{cases} H(x) \\ A(x) \neq 0 \end{cases}$ no solution in \mathbb{R}^k

the incompatibilities above, the one below.

If we know how to do that, we say we have a weak inference:

$$A \neq 0 \vdash A < 0 \vee A > 0$$

$$\downarrow H, A < 0 \downarrow \rightarrow \downarrow H, A \neq 0, -A \geq 0 \downarrow \rightarrow S_1 A^{2c_1} + N_1 + Z_1 = AM_1$$

$$\downarrow H, A > 0 \downarrow \rightarrow \downarrow H, A \neq 0, A \geq 0 \downarrow \rightarrow S_2 A^{2c_2} + N_2 + Z_2 = -AM_2$$

$$\underbrace{S_1 S_2 A^{2c_1+2c_2}}_S + \underbrace{\left(N_1 (S_2 A^{2c_2} + N_2) + S_1 A^{2c_1} N_2 + A^2 M_1 M_2 \right)}_N + \underbrace{\left(Z_1 (S_2 A^{2c_2} + N_2 + Z_2) + Z_2 (S_1 A^{2c_1} + N_1) \right)}_Z = 0$$

More interesting weak inferencer

→) $P \in K[x_1, \dots, x_n, y]$, monic in y , odd degree in y

$$\vdash \exists y / P(x, y) = 0.$$

→) $P \in K[x_1, \dots, x_n, z]$, monic in z

$$\vdash \exists z = a + ib / \underbrace{P(x, z)} = 0$$

$$P_{Re}(x, a, b) = P_{Im}(x, a, b) = 0.$$

In general: given $P \subseteq K[x_1, \dots, x_n]$

(6)

$$P_k = P \subseteq K[x_1, \dots, x_n]$$

$$P_{k-1} = \text{Elim}(P_k) \subseteq K[x_1, \dots, x_{n-1}]$$

$$P_1 = \text{Elim}(P_2) \subseteq K[x_1]$$

$$P_0 = \text{Elim}(P_1) \subseteq K.$$

We produce for every realizable sign condition σ on P_i ,

$$\text{sign}(P_i) = \sigma \vdash \bigvee \text{sign}(P_{i+1}) = \sigma',$$

σ' realizable sign condition on P_{i+1}
given that $\text{sign}(P_i) = \sigma$

Finally, given a system \mathcal{H} with no solutions in \mathbb{R}^n , we take $P =$
 $= \{P_1, P_{n-1}, P_1, P_{n-2}, P_1, P_{n-3}, \dots, P_{n-1}\}$ and for every realizable sign condition σ
on $P_k = P$, we produce easily an inconsistency

$$\downarrow \text{sign}(P_k) = \sigma, \mathcal{H} \downarrow.$$

After "eliminating" all variables, we obtain for the only realizable
sign condition σ on P_0 , an inconsistency

$$\downarrow \text{sign}(P_0) = \sigma, \mathcal{H} \downarrow \longrightarrow \downarrow \mathcal{H} \downarrow \checkmark$$