Lebesgue motivic invariants

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1. Motivation

Zeta function of a real analytic function:

Let $n \in \mathbb{N}$ and let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a real analytic function. For $k \in \mathbb{N}$ let

$$\mathcal{L}_k := \mathcal{L}_k(\mathbb{R}^n, 0) = \left\{\gamma(t) = a_1 t + a_2 t^2 + \ldots + a_k t^k \mid a_1, \ldots, a_k \in \mathbb{R}^n\right\}$$

denote the space of polynomial arcs of degree at most k vanishing at the origin and

$$X_k(f) := \left\{\gamma(t) \in \mathcal{L}_k \mid \mathrm{ord}(f \circ \gamma) = k\right\}.$$

The **zeta function** of f is defined by

$$Z_f(T) := \sum_{k=1}^{\infty} (-1)^{-kn} \chi\big(X_k(f)\big) T^k \in \mathbb{Z}[[T]]$$

where χ denotes the Euler characteristic.

- Koike, Parusinski
- Denef, Loeser
- Fichou

Arc space

Motivic invariants

2. Integration on the field of Puiseux series

Let

$$\mathbb{P} := \Big\{ \sum_{j=k}^{\infty} a_j t^{j/p} \mid p \in \mathbb{N}, k \in \mathbb{Z}, (a_j) \subset \mathbb{R} \Big\}$$

be the field of Puiseux series over $\mathbb{R}$.

Valuation:

The map

$$v : \mathbb{P} \to \mathbb{Q} \cup \{\infty\}, f \mapsto \min \operatorname{supp}(f),$$

is a valuation on $\mathbb{P}$. Its valuation ring is the convex hull of $\mathbb{Z}$ and is given by

$$\mathcal{O} := \{f \in \mathbb{P} \colon \operatorname{supp}(f) \subset \mathbb{Q}_{\geq 0}\}.$$

Its maximal ideal is given by

$$\mathfrak{m} := \{f \in \mathbb{P} \colon \operatorname{supp}(f) \subset \mathbb{Q}_{>0}\}.$$

- $\mathbb{P}$ is a model of the theory T_{an} of the structure $\mathbb{R}_{\mathrm{an}}$ (sets and functions definable in $\mathbb{R}_{\mathrm{an}}$ are precisely the globally subanalytic sets and functions).
- $\mathbb{P}$ carries a partial logarithm

$$\log : \mathbb{R}_{>0} + \mathfrak{m} \to \mathbb{R} + \mathfrak{m}, a(1+h) \mapsto \log(a) + L(h),$$

where $a \in \mathbb{R}_{>0}, h \in \mathfrak{m}$ and $L(x) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} x^j$ denotes the logarithmic series.

- The field of Puiseux series is a subfield of the field $\mathbb{T}$ of transseries. The partial logarithm on $\mathbb{P}$ can be extended to a logarithm

$$\log : \mathbb{P}_{>0} \to \mathbb{T}, at^q(1+h) \mapsto \log(a) + L(h) - qX,$$

where $a \in \mathbb{R}_{>0}, h \in \mathfrak{m}, q \in \mathbb{Q}$ and $X := \log(t^{-1}) \in \mathbb{T} \setminus \mathbb{P}$. Note that X is transcendental over $\mathbb{P}$ with $\mathcal{O} < X < \mathbb{P}_{>0} \setminus \mathcal{O}$.

I have established a complete Lebesgue measure and integration theory for the category of globally subanalytic sets and functions over $\mathbb{P}$:

$$\{\text{gl. subanal. subsets of } \mathbb{P}^n\} \to \mathbb{P}[X] \cup \{\infty\},$$

$$A \mapsto \lambda_{\mathbb{P},n}(A),$$

and

$$\{\text{integrable gl. subanal. fcts } \mathbb{P}^n \to \mathbb{P}\} \to \mathbb{P}[X],$$

$$f \mapsto \int f\, d\lambda_{\mathbb{P},n},$$

such that the usual properties hold (adjusted to the globally subanalytic setting):

The measure is

- additive,
- monotone,
- reflects elementary geometry.

The integral fulfills

- the transformation theorem,
- Lebesgue's theorem on dominated convergence.

Extension of the valuation:
The valuation v is extended to a valuation v^* on $\mathbb{P}[X]$ as follows: The value group of v^* is given by $\mathbb{Q} \times \mathbb{Z}$ equipped with the lexicographical order. We have $v^*(X) = (0, -1)$.

3. Lebesgue zeta series

Let $n \in \mathbb{N}$ and let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be real analytic such that 0 is an **isolated zero**.
For $q \in \mathbb{Q}_{>0}$ let

$$Y_q(f) := \{x \in \mathbb{P}^n \mid v(|x|) \geq 1/q \text{ and } v(f(x)) \geq q\}$$

and

$$\mathcal{Y}_q(f) := \{A \subset \mathbb{P}^n \text{ gl. suban.} \mid A \subset Y_q(f)\}.$$

Proposition:
For every $q \in \mathbb{Q}_{>0}$ and every $A \in \mathcal{Y}_q(f)$ we have that

$$v^*\big(\lambda_{\mathbb{P},n}(A)\big) \in \mathbb{Q}_{>0} \times \{-n+2, \ldots, 0\}.$$

Proposition:
For every $q \in \mathbb{N}$ the function

$$\mathcal{Y}_q(f) \to \mathbb{Q} \times \mathbb{Z}, A \mapsto v^*\big(\lambda_{\mathbb{P},n}(A)\big),$$

attends a minimum.

We denote this minimum by $\mu_q(f)$.

We set

$$LZ_f(T) := \sum_{k\in\mathbb{N}} \mu_k(f)T^k \in (\mathbb{Q}\times\mathbb{Z})[[T]]$$

and call it the **Lebesgue zeta series** of f.

Remark:
Let $n \leq 2$. Then $LZ_f(T) \in \mathbb{Q}[[T]]$.

Example
Let $f : \mathbb{R} \to \mathbb{R}$ be the identity. Then

$$LZ_f(T) = \sum_{k=1}^{\infty} kT^k = -\frac{T}{(1-T)^2}.$$

Two globally subanalytic functions $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are globally subanalytic bi-Lipschitz equivalent if there is a globally subanalytic bi-Lipschitz map $H : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f = g \circ H$.

Theorem:
The Lebesgue zeta series is invariant under globally subanalytic bi-Lipschitz equivalence.

Let $\mathcal{Q} = \mathbb{Q} \cup \{\infty\}$ be the **tropical semiring**:

$$\oplus : \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}, (a, b) \mapsto \min\{a, b\},$$

$$\otimes : \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}, (a, b) \mapsto a + b.$$

We have that $0_{\mathcal{Q}} = \infty$ and $1_{\mathcal{Q}} = 0_{\mathbb{Q}}$.

As usually, the polynomial semiring $\mathcal{Q}[T]$ and the semiring $\mathcal{Q}[[T]]$ of power series over $\mathcal{Q}$ are defined.
For $f = \sum_{k=1}^{\infty} a_k T^k \in \mathcal{Q}[[T]]$ the power series

$$f^* := \sum_{k=0}^{\infty} f^{\otimes k}$$

is called the **star** of f.

Example: $(1_{\mathbb{Q}} T)^* = \sum_{k=0}^{\infty} kT^k$.

The semiring of **rational series** is the smallest subsemiring S of $\mathcal{Q}[[T]]$ with the following properties:

- $\mathcal{Q}[T] \subset S$,
- $f \in S$ with $f(0) = 0_{\mathcal{Q}} \Longrightarrow f^* \in S$.

Theorem:
Let $n \leq 2$. Then the Lebesgue zeta function $LZ_f(T) \in \mathcal{Q}[[T]]$ is rational.