Orderings and $\mathbb{R}$-places of function fields

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Space of orderings and $\mathbb{R}$ - places

For any formally real field K we denote:

$X(K)$ - the space of orderings of K with the Harrison topology,

$M(K)$ - the space of $\mathbb{R}$-places of K with the quotient topology inherited from the space $X(K)$.

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$$X(K(x)) \quad \rightleftarrows \quad \mathcal{C}(K)$$

Harrison topology — homeomorphism — order topology

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For $B = B_S(a)$ define:

B^- the cut in K with the lower cut set $\{a \in K : a < B\}$,

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The cuts defined above we call ball cuts.

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Theorem (F.-V. Kuhlmann, M. Machura, K. K., 2010)

Let $C_1 < C_2$ *be cuts in* K*. The corresponding orderings of* $K(x)$ *determine the same* $\mathbb{R}$*-place iff* $C_1 = B^-$ *and* $C_2 = B^+$ *for some ultrametric ball* B *in* K*.*

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Let $C_1 < C_2$ be cuts in K. The corresponding orderings of $K(x)$ determine the same $\mathbb{R}$-place iff $C_1 = B^-$ and $C_2 = B^+$ for some ultrametric ball B in K.

If the field K is Archimedean, then we obtain a topological circle as the space of $\mathbb{R}$-places of $K(x)$.

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This implies that the space $M(K(x))$ carries a lot of self-similarities.

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How do the orderings of the function field F correspond to the structure of the curve γ?

The structure of a real algebraic curve over K

M. Knebusch *On algebraic curves over real closed fields* (I and II)

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$$\mathcal{H}(F \mid K) = \{f \in F \mid f(p) \neq \infty \text{ for every } p \in \gamma\}$$

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Let $\gamma_1, ..., \gamma_n$ be the distinct equivalence classes.

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For every γ_i there is a function $\eta_i \in F$ such that

$$sgn(\eta_i(p)) = \begin{cases} -1 & \text{if } p \in \gamma_i \\ 1 & \text{if } p \notin \gamma_i \, . \end{cases}$$

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The function η_i is determined uniquely up to multiplication by SOS.

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For every interval $(p,q) \subset \gamma_i$ there is a function $\chi_{(p,q)} \in F$ such that

$$sgn(\chi_{(p,q)}(r)) = \begin{cases} -1 & \text{if } r \in (p,q) \\ 0 & \text{if } r \in \{p,q\} \\ 1 & \text{if } r \notin [p,q]\,. \end{cases}$$

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The function $\chi_{(p,q)}$ is called an interval function for (p,q) and it is determined uniquely up to SOS.

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Theorem

For every $f \in F$ and every γ_i there is a finite number of points $p_1 < ... < p_n$ on γ_i such that f is definite and monotonic on the intervals $(p_1, p_2), ..., (p_{n-1}, p_n), (p_n, p_1)$.

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Theorem

Every cut (L, U) of γ_i corresponds to some ordering of F.

$$P_{(L,U)} = \{f \in F \mid \exists\, l \in L \cup \{\infty_i\}\, \exists\, u \in U \cup \{\infty_i\}\, \forall\, p \in (l, u) : f(p) > 0\}.$$

Cuts on γ and orderings of F

Principal cuts for $p \in \gamma_i$:

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Denote

$$X_{princ}(F) = \{P_{p^-}, P_{p^+} \mid p \in \gamma\}.$$

Theorem (A. Prestel, *Lectures on Formally Real Fields*, Th.9.9.)

$X_{princ}(F)$ is dense in $X(F)$.

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Assume $-\eta_i \in P$ and $-\eta_j \in P$. Then the Harrison basic set $H(-\eta_i) \cap H(-\eta_j)$ is nonempty.

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This contradiction proves:

Lemma

For every $P \in X(F)$ there is exactly one Knebusch component γ_i such that $-\eta_i \in P$.

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This component we call associated with the ordering P.

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Then (L, U) is a cut on γ_i and $P_{(L,U)} = P$.

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We obtain a bijection

$$b : X(F) \to \mathcal{C}(\gamma) .$$

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$C(\gamma)$ carries the topology of the union of cyclically ordered sets. Take $C_1 < C_2 \in C(\gamma_i)$. Then

$$b^{-1}((C_1, C_2)) = \bigcup_{p,q \in U_1 \cap L_2} H(-\chi(p,q))\,.$$

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Theorem

The space of orderings of F is homeomorphic to the space of cuts of γ.

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and the following commuting diagram:

Cuts on γ and orderings of F

$$\begin{array}{ccc} \mathcal{C}(\gamma) & \longrightarrow & X(F) \\ \downarrow{\scriptstyle res_x} & & \downarrow{\scriptstyle res} \\ \mathcal{C}(K) & \longrightarrow & X(K(x)) \, . \end{array}$$

where horizontal maps are homeomorphisms.

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Fact

If $(L, v)/(K, v)$ is a finite extension of valued fields then

$$[vL : 2vL] = [vK : 2vK].$$

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Definition

A cut C of γ is called a ball cut if $res_x(C)$ is a ball cut for every $x \in F \setminus K$.

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Assume that we have chosen proper coordinates and we have an embedding of γ in K^m. The vector space K^m is an ultrametric space with the ultrametric distance:

$$d(p,q) = \min\{v(p_1 - q_1), ..., v(p_m - q_m)\} = \frac{1}{2} v(\sum(p_i - q_i)^2),$$

for $p = (p_1, .., p_m)$ and $q = (q_1, ..., q_m)$.

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Theorem

Every ball cut on γ is induced by some ultrametric ball in K^m.

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Assume that the corresponding orderings P_1 and P_2 of F determine different $\mathbb{R}$-places: $\lambda(P_1)$ and $\lambda(P_2)$.

Then there is $x \in F \setminus K$ such that $\lambda(P_1)(x) > 0$ and $\lambda(P_2)(x) < 0$.

Thus $\lambda(P_1)\mid_{K(x)} \neq \lambda(P_2)\mid_{K(x)}$ and therefore $res_x(C_1) \nsim res_x(C_2)$.

Theorem

Let C_1 and C_2 be two ball cuts on γ. The corresponding orderings determine the same $\mathbb{R}$-place of F iff for every $x \in F \setminus K$ the cuts $res_x(C_1)$ and $res_x(C_2)$ are ball cuts of the same ultrametric ball.

Thank you very much for your attention!