Dp-minimal ordered fields

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joint work with Pierre Simon (Lyon 1) and Erik Walsberg (UCLA)

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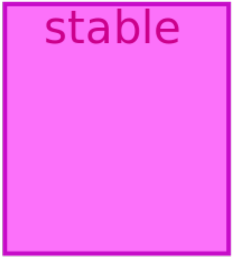
The realm of neostability



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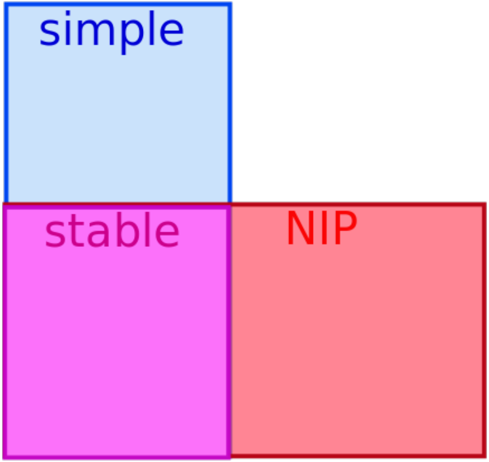

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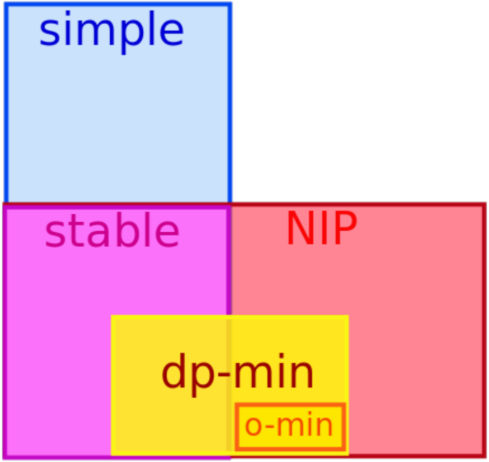

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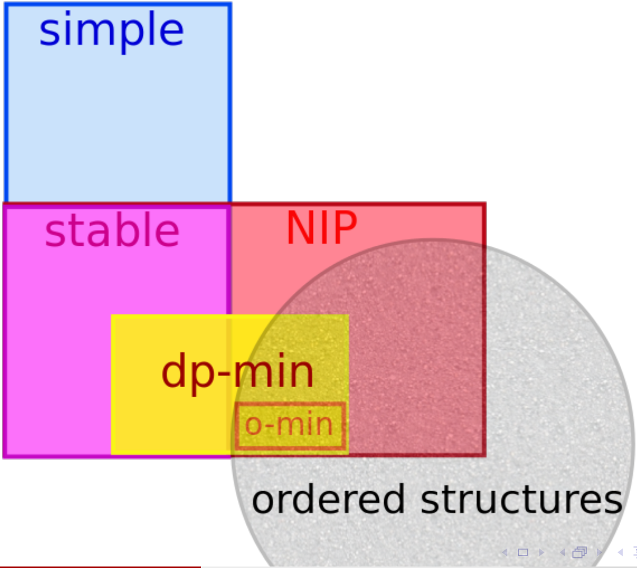

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Example: Any o-minimal theory is dp-minimal. In particular, the $\mathcal{L}_{\mathrm{ring}}$-theory of $\mathbb{R}$ is dp-minimal.

Henselian fields and dp-minimality

Fact (Chernikov-Simon)

Let (K, v) be a henselian valued field of equicharacteristic 0. Then

$$\mathrm{Th}(K, v) \text{ is dp-minimal} \iff \mathrm{Th}(Kv) \text{ and } \mathrm{Th}(vK) \text{ are dp-minimal.}$$

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We get: If $[\Gamma : p\Gamma]$ is finite for all p, then $\mathrm{Th}(\mathbb{R}((\Gamma)))$ is dp-minimal (in $\mathcal{L}_{\mathrm{val}}$ and hence also in $\mathcal{L}_{\mathrm{ring}}$ and $\mathcal{L}_{\mathrm{of}}$).

Ordered fields, dp-minimality and definable valuations

Fact (Guingona)

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(See board for proof.)

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Thank you for your attention!

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Dp-minimal fields

Theorem (Will Johnson):

1. Let (K, v) be a henselian defectless valued field with residue field k and value group Γ. Suppose
   - $k \models \mathrm{ACF}_p$ or k is elementarily equivalent to a local field of characteristic 0.
   - for every n, $[\Gamma : n\Gamma]$ is finite.
   - if k has characteristic p, and $\gamma \in [-v(p), v(p)]$ then $\gamma \in p\Gamma$. Here $[-v(p), v(p)]$ denotes Γ if K has characteristic p.

   Then (K, v) is dp-minimal as a valued field, and the theory of (K, v) is completely determined by the theories of k and Γ (or k and $(\Gamma, v(p))$ in mixed characteristic).
2. Let K be a sufficiently saturated dp-minimal field. Then there is some valuation on K satisfying the conditions above.