Supertropical algebra and representations

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A **simplicial complex (s.c.)** is a pair $\mathcal{S} := (E, \mathcal{H})$, with E a finite set and $\mathcal{H} \subseteq \mathrm{Pw}(E)$, that satisfies the axioms:

A. $\mathcal{H}$ is nonempty,

B. $Y \subseteq X$, $X \in \mathcal{H} \;\Rightarrow\; Y \in \mathcal{H}$.

A **basis** is a maximal simplex (with respect to inclusion).

A **matroid** $\mathcal{M} := (E, \mathcal{H})$ is s.c. that admits the extra axiom:

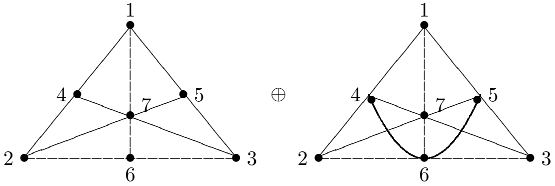
EX. If $X, Y \in \mathcal{H}$ and $|X| = |Y| + 1$, then there exists $x \in X \setminus Y$ such that $Y \cup \{x\}$ is in $\mathcal{H}$.

A **realizations** of a s.c. is an embedding $\varphi : E \to \mathcal{M}$, mapping E to elements of a module $\mathcal{M}$, which respects independence:

$$\varphi(X) \text{ is (linearly) independent} \quad \Leftrightarrow \quad X \in \mathcal{H}, \qquad \forall X \subseteq E.$$

A matroid $\mathcal{M}$ is **field-realizable** if it has a realization by a vector space; $\mathcal{M}$ is **regular** if it is realizable over any field.

Not all matroids are field-realizable, for example the direct sum $F^- \oplus F$



of the non-Fano and the Fano matroid is not field-realizable.

Tropical mathematics

A **semiring** $(R, +, \cdot)$ is a structure such that $(R^{\times}, \cdot)$ is a monoid and $(R, +)$ is a commutative monoid, with distributivity of multiplication over addition on both sides.

Tropical mathematics is customarily developed over the **max-plus semiring** $(\overline{\mathbb{R}}, +, \cdot)$, $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$, whose addition and multiplication are maximum and summation, respectively:

$$a + b := \max\{a, b\}, \quad a \cdot b := a \underset{\text{sum}}{+} b,$$

where $\mathbb{0} := -\infty$, $\mathbb{1} := 0$.

- lack of additive inverse,
- $\overline{\mathbb{R}}$ is idempotent, i.e. $a + a = a$ for any a.

Combinatorial approach

The notion of "vanishing" of an equation

$$q := q_1 + q_2 + \cdots + q_m$$

is replaced by taking elements on which the maximum of q is attained simultaneously by at least two different terms.

For example, a **tropical hypersurface** is the **corner locus** of a tropical polynomial

$$f := \sum_{\mathbf{i} \in \Omega} \alpha_{\mathbf{i}} \lambda_1^{i_1} \cdots \lambda_m^{i_m},$$

i.e., the domain of nonsmoothness the convex piecewise linear function $\tilde{f} : \mathbb{R}^{(m)} \to \mathbb{R}$:

$$\tilde{f}(a_1, \dots, a_m) = \max_{\mathbf{i} \in \Omega} \{ i_1 a_1 \underset{\text{sum}}{+} \cdots \underset{\text{sum}}{+} i_m a_m \underset{\text{sum}}{+} \alpha_{\mathbf{i}} \}.$$

Supertropical algebra

A **supertropical semiring** is a semiring $R := (R, \mathcal{G}_{\mathbb{0}}, \nu)$ with:

- a distinguished ideal $\mathcal{G}_{\mathbb{0}}$, called the **ghost ideal**, and
- a semiring projection $\nu : R \to \mathcal{G}_{\mathbb{0}}$, called the **ghost map**,

satisfying the axioms (writing a^ν for $\nu(a)$):

$$\begin{array}{rlll} \text{Supertropicality:} & a + b = a^\nu & \text{if} & a^\nu = b^\nu; \\ \text{Bipotence:} & a + b \in \{a, b\} & \text{if} & a^\nu \neq b^\nu. \end{array}$$

A **supertropical semifield** $F := (F, \mathcal{G}_{\mathbb{0}}, \nu)$ is a supertropical semiring for which:

- $\mathcal{T} := F \setminus \mathcal{G}_{\mathbb{0}}$ is an Ablian group (called the **tangible part**);
- the restriction $\nu|_{\mathcal{T}} : \mathcal{T} \to \mathcal{G}$ is onto.

Suppose $G = (G, *, <)$ is an abelian ordered group, s.t. $ca > cb \Rightarrow a > b$.
Define the set $F := G \cup \{\mathbb{0}\} \cup G^\nu$, ordered as

$$a^\nu \ >_\nu \ a \ >_\nu \ b^\nu \ >_\nu \ b \ >_\nu \ \mathbb{0}, \qquad \text{for any } a > b \text{ in } G.$$

Set $\mathcal{G}_\mathbb{0} := G^\nu \cup \{\mathbb{0}\}$, and let $\nu : F \to \mathcal{G}_\mathbb{0}$ be the ghost map given by $a \mapsto a^\nu$.
$(F, \mathcal{G}_\mathbb{0}, \nu)$ is a supertropical semifield with operations $(x, y \in F)$:

- $x + y := \begin{cases} \max\{x, y\} & \text{if } x^\nu \neq y^\nu \\ x^\nu & \text{else} \end{cases}$
- $a \cdot b := a * b, \qquad a^\nu \cdot b = a \cdot b^\nu = a^\nu \cdot b^\nu := (a * b)^\nu,$
  $\mathbb{0} \cdot x = x \cdot \mathbb{0} = \mathbb{0}.$

The **superboolean semifield** $\mathbb{SB}$ is the finite supertropical semifield defined over $\{1, 0, 1^\nu\}$, equipped with the total order

$$1^\nu >_\nu 1 >_\nu 0,$$

and endowed with the binary operations

$+$	0	1	1^ν
0	0	1	1^ν
1	1	1^ν	1^ν
1^ν	1^ν	1^ν	1^ν

$\cdot$	0	1	1^ν
0	0	0	0
1	0	1	1^ν
1^ν	0	1^ν	1^ν

that modify the standard operations of the boolean semiring $(\mathbb{B}, \wedge, \vee)$.

Rmk. $\mathbb{SB}$ *provides a type of* 3*-value logic, associated with a commutative associative algebra.*

Philosophy

A supertropical semiring is not idempotent, i.e.

$$a + a = a + \cdots + a = a^{\nu}.$$

Along all our development:

Classical world :	$\rightsquigarrow$	Supertropical world:
$\square \cdots \square = 0$		$\square \cdots \square \in \mathcal{G}_{\mathbb{0}}$

Namely, the ghost ideal $\mathcal{G}_{\mathbb{0}}$ plays the role of the zero element in classical mathematics.

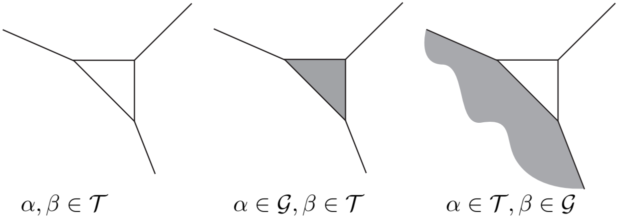
Def. *The* **root set** *of* $f \in F[\lambda_1, \dots, \lambda_n]$ *is defined as*

$$Z(f) = \{a = (a_1, \dots, a_n) \in F^{(n)} \mid f(a) \in \mathcal{G}_{\mathbb{0}}\},$$

a root $a \in \mathcal{T}_{\mathbb{0}}^{(n)}$ *is called* **tangible**.

The geometry associated to this theory is polyhedral geometry.

Ex. $Z(f) \cap \mathcal{T}^{(2)}$ *of* $f = \lambda_1^2\lambda_2 + \lambda_1\lambda_2^2 + \alpha\lambda_1\lambda_2 + \beta \in \mathbb{T}[\lambda_1, \lambda_2]$*:*



Tangible roots of tangible polynomials correspond to the corner loci of polynomials over the max-plus algebra.

This approach provides new examples of algebraic sets which were previously inaccessible such as algebraic subsets of codimension 0.

Matrices and digraphs

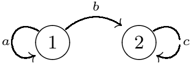
Matrices over semifields are adjacency matrices of digraphs:

Digraphs

Boolean matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

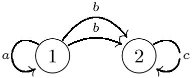
Weighted digraphs



Max-plus matrices

$$\begin{pmatrix} a & b \\ \mathbb{0} & c \end{pmatrix}$$

Weighted digraph + double edges



Supertropical matrices

$$\begin{pmatrix} a & {\color{red} b^{\nu}} \\ \mathbb{0} & c \end{pmatrix}$$

Over this setting, algebraic notions take combinatorial meanings and digraphs are a major computational tool in tropical matrix theory.

Matrix algebra

The **permanent** of a matrix $A = (a_{i,j})$ is defined as

$$\mathrm{per}(A) = \sum_{\pi \in S_n} a_{1,\pi(1)} \cdots a_{n,\pi(n)}.$$

The **minor** $A'_{i,j}$ is obtained by deleting the i row and j column of A.
The **adjoint** matrix $\mathrm{adj}(A)$ is the transpose of the matrix $(a'_{i,j})$, where $a'_{i,j} = \mathrm{per}(A'_{i,j})$.

Def. *A matrix A is* **nonsingular** *if $\mathrm{per}(A)$ is tangible; otherwise, when $\mathrm{per}(A) \in \mathcal{G}_{\mathbb{0}}$, A is called* **singular**.

The permanent is not multiplicative!

Ex. Take the nonsingular matrix

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \quad \textit{for which} \quad A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Then $\operatorname{per}(A)^2 = 2^2 = 4$, *while* $\operatorname{per}(A^2) = 5^\nu$. *So* $\operatorname{per}(AB) \neq \operatorname{per}(A)\operatorname{per}(B)$.

Thm. For any matrices A, B over a supertropical semifield

$$\operatorname{per}(AB) = \operatorname{per}(A)\operatorname{per}(B) + ghost;$$

namely $\operatorname{per}(AB) \geq_\nu \operatorname{per}(A)\operatorname{per}(B)$, *where* $\operatorname{per}(AB) = \operatorname{per}(A)\ \operatorname{per}(B)$ *whenever AB is nonsingular.*

Def. *A subset* $W = \{\mathbf{v}_1, \dots, \mathbf{v}_m\} \subset F^{(n)}$ *is* **dependent** *if there is a finite sum* $\sum \alpha_i \mathbf{v}_i \in \mathcal{G}_{\mathbb{0}}^{(n)}$*, with each* $\alpha_i \in \mathcal{T}_{\mathbb{0}}$*, but not all* $\mathbb{0}$*; otherwise* W *is called* **independent***.*

Tropical dependence does not coincide with spanning; for example the vectors

$$\mathbf{v}_1 = (\mathbb{1}, \mathbb{1}, \mathbb{0}), \quad \mathbf{v}_2 = (\mathbb{1}, \mathbb{0}, \mathbb{1}), \quad \text{and} \quad \mathbf{v}_3 = (\mathbb{0}, \mathbb{1}, \mathbb{1}),$$

are dependent, i.e. $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 \in \mathcal{G}_{\mathbb{0}}^{(3)}$, but none of them can be written in terms of the others.

Def. *The* **row (column) rank** *of a matrix A is the maximal number of independent rows (column) of A.*

Thm. *The rank of a matrix A is equal the maximal k such that A has a nonsingular $k \times k$ submatrix.*

Cor. *A matrix is nonsingular iff its rows (columns) of are independent.*

Cor. *Any $n + 1$ vectors in $F^{(n)}$ are dependent.*

Def. *A* **quasi-identity** *matrix* $\mathcal{I}$ *is a nonsingular (multiplicatively) idempotent matrix, i.e.* $\mathcal{I} = \mathcal{I}^2$.

Accordingly, $\mathrm{per}(\mathcal{I}) = \mathbb{1}$, the diagonal entries of $\mathcal{I}$ are all $\mathbb{1}$, while $\mathcal{I}$ is ghost off the diagonal. (The identity matrix I is clearly quasi-identity.)

Def. *A matrix* B *is a* **quasi-inverse** *of a matrix* A *if both* AB *and* BA *are quasi-identities;* A *is* **quasi-invertible** *if it has a quasi-inverse.*

Thm. *A matrix* A *is quasi-invertible iff* A *is nonsingular, in this case* $A^{\nabla} := \frac{\mathrm{adj}(A)}{\mathrm{per}(A)}$ *is the* **canonical quasi-inverse** *of* A.

Realizations of simplicial complexes

Recall that the superboolean semifield is the finite supertropical semifield $\mathbb{SB} := \{0, 1, 1^\nu\}$.

Any $m \times n$ supertropical matrix A generates a simplicial complex $\mathcal{H}(A) := (\mathrm{col}(A), \mathcal{H}(A))$ whose simplices are determined by the independent columns of A.

Ex. *The matrix*

$$A := \begin{pmatrix} 1 & 0 & 1 & 1^\nu \\ 0 & 1 & 1 & 1 \\ \hline a & b & c & d \end{pmatrix}$$

determines a simplicial complex (which is not a matroid).

A **superboolean-realization** of a s.c. $\mathcal{S} := (E, \mathcal{H})$ is a bijective map $\varphi : E \to \mathrm{col}(A)$ that respects simplices.

Thm. *Any simplicial complex is superboolean-representable.*

Lem. *A $k \times k$ matrix $W_k \in M_k(\mathbb{SB})$ is nonsingular iff by independently permuting columns and rows it can be rearranged to the triangular form*

$$A' := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ * & \cdots & * & 1 \end{pmatrix}.$$

A $k \times k$ nonsingular submatrix W_k is called k-**witness**. A k-**marker** is a row of length k having a single 1-entry and all its other entries are 0.

- Any k-witness contains a k-marker.
- Any subset of k independent columns contains a k-witness.

Proof. *Naive construction of a superboolean realization.*

Let $\mathcal{B}_1, \dots, \mathcal{B}_m$, $|\mathcal{B}_i| := k_i$, be the bases of $\mathcal{S} := (E, \mathcal{H})$.

- Start with a $k_1 \times n$ matrix

$$A_1 := \underbrace{\begin{array}{|c|c|} \mathrm{W}_{k_1}(\mathcal{B}_1) & (1^\nu) \\ \hline \end{array}}_{\mathcal{B}_1}$$

  whose k_1 left columns are labeled by $\mathcal{B}_1$.

- Reorder the columns of A_1 such that $\mathcal{B}_2$ corresponds to the k_2 left columns, pile a k_2-witness on the left, and let the other entries be 1^ν

$$A_2 := \underbrace{\begin{array}{|c|c|} \mathrm{W}_{k_2}(\mathcal{B}_2) & (1^\nu) \\ \hline \multicolumn{2}{|c|}{A_1 \text{ “reordered”}} \\ \hline \end{array}}_{\mathcal{B}_2}$$

- Repeat this process for each basis $\mathcal{B}_i$, $i = 3, \dots, m$.

Given a supertropical semifield F, there is a natural embedding $\varphi : \mathbb{SB} \to F$:

$$\varphi : 1 \mapsto \mathbb{1}, \quad \varphi : 1^\nu \mapsto \mathbb{1}^\nu, \quad \varphi : 0 \mapsto \mathbb{0}.$$

Cor. Every s.c. is "super regular", i.e. it is F-realizable over any supertropical semifield F.

The superboolean framework allows also realization of posets, lattices, and quivers.

Thm. Any matroid is boolean (tangible) realizable, and hence also tropical realizable.