The analogue of Hilbert's 1888 Theorem for even symmetric forms

Charu Goel
University of Konstanz

(Joint with Salma Kuhlmann and Bruce Reznick)

Ordered Algebraic Structures and Related Topics
CIRM - Luminy, France

October 13th, 2015

Outline

Outline

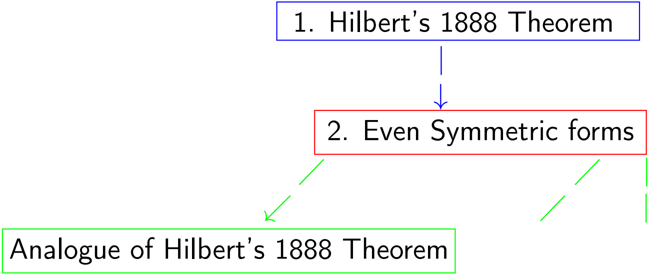
Outline

1. Hilbert's 1888 Theorem

↓

2. Even Symmetric forms

Outline


1. Hilbert's 1888 Theorem
2. Even Symmetric forms
Analogue of Hilbert's 1888 Theorem

Outline

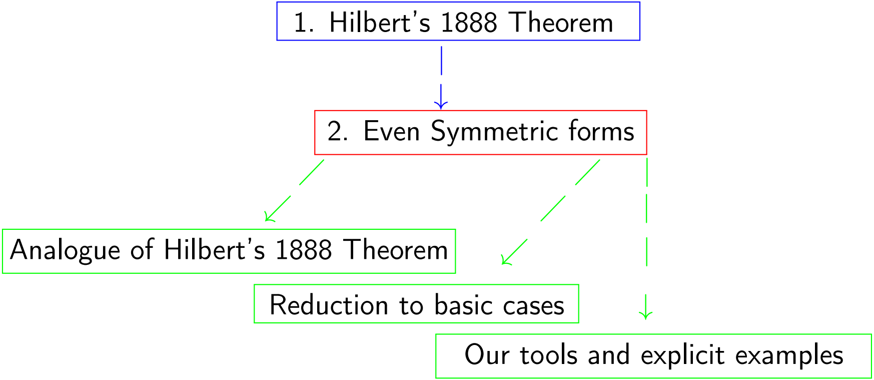
1. Hilbert's 1888 Theorem

2. Even Symmetric forms

Analogue of Hilbert's 1888 Theorem

Reduction to basic cases

Outline


1. Hilbert's 1888 Theorem
2. Even Symmetric forms
Analogue of Hilbert's 1888 Theorem
Reduction to basic cases
Our tools and explicit examples

1. Hilbert's 1888 Theorem

- Let $\mathcal{P}_{n,2d}$, $\Sigma_{n,2d}$ be the set of n-ary $2d$-ic forms that are **positive semidefinite (psd)**, respectively **sum of squares (sos)**.

1. Hilbert's 1888 Theorem

- Let $\mathcal{P}_{n,2d}$, $\Sigma_{n,2d}$ be the set of n-ary $2d$-ic forms that are **positive semidefinite (psd)**, respectively **sum of squares (sos)**.
- **Theorem (Hilbert, 1888):** $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ if and only if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.

1. Hilbert's 1888 Theorem

- Let $\mathcal{P}_{n,2d}$, $\Sigma_{n,2d}$ be the set of n-ary $2d$-ic forms that are **positive semidefinite (psd)**, respectively **sum of squares (sos)**.
- **Theorem (Hilbert, 1888):** $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ if and only if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.
  - The arguments for $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ for $n = 2$ and $2d = 2$ were already known in the late 19th century.

1. Hilbert's 1888 Theorem

- Let $\mathcal{P}_{n,2d}$, $\Sigma_{n,2d}$ be the set of n-ary $2d$-ic forms that are **positive semidefinite (psd)**, respectively **sum of squares (sos)**.
- **Theorem (Hilbert, 1888):** $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ if and only if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.
  - The arguments for $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ for $n = 2$ and $2d = 2$ were already known in the late 19th century.

    Hilbert proved $\mathcal{P}_{3,4} = \Sigma_{3,4}$.

1. Hilbert's 1888 Theorem

- Let $\mathcal{P}_{n,2d}$, $\Sigma_{n,2d}$ be the set of n-ary $2d$-ic forms that are **positive semidefinite (psd)**, respectively **sum of squares (sos)**.
- **Theorem (Hilbert, 1888):** $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ if and only if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.
  - The arguments for $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ for $n = 2$ and $2d = 2$ were already known in the late 19th century.

    Hilbert proved $\mathcal{P}_{3,4} = \Sigma_{3,4}$.
  - Conversely,

1. Hilbert's 1888 Theorem

- Let $\mathcal{P}_{n,2d}$, $\Sigma_{n,2d}$ be the set of n-ary $2d$-ic forms that are **positive semidefinite (psd)**, respectively **sum of squares (sos)**.
- **Theorem (Hilbert, 1888):** $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ if and only if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.
  - The arguments for $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ for $n = 2$ and $2d = 2$ were already known in the late 19th century.

    Hilbert proved $\mathcal{P}_{3,4} = \Sigma_{3,4}$.
  - Conversely, Hilbert proved that $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$ and $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$, and demonstrated that it is enough for all remaining cases, i.e.

1. Hilbert's 1888 Theorem

- Let $\mathcal{P}_{n,2d}$, $\Sigma_{n,2d}$ be the set of n-ary $2d$-ic forms that are **positive semidefinite (psd)**, respectively **sum of squares (sos)**.
- **Theorem (Hilbert, 1888):** $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ if and only if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.
  - The arguments for $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ for $n = 2$ and $2d = 2$ were already known in the late 19th century.

    Hilbert proved $\mathcal{P}_{3,4} = \Sigma_{3,4}$.
  - Conversely, Hilbert proved that $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$ and $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$, and demonstrated that it is enough for all remaining cases, i.e.

    Proposition [Reduction to Basic cases]:
    If $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$ and $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$, then $\Sigma_{n,2d} \subsetneq \mathcal{P}_{n,2d}$ for all $n \geq 3, 2d \geq 4$ and $(n, 2d) \neq (3, 4)$.

1. Hilbert's 1888 Theorem

- Let $\mathcal{P}_{n,2d}$, $\Sigma_{n,2d}$ be the set of n-ary $2d$-ic forms that are **positive semidefinite (psd)**, respectively **sum of squares (sos)**.
- **Theorem (Hilbert, 1888):** $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ if and only if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.
  - The arguments for $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ for $n = 2$ and $2d = 2$ were already known in the late 19th century.

    Hilbert proved $\mathcal{P}_{3,4} = \Sigma_{3,4}$.
  - Conversely, Hilbert proved that $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$ and $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$, and demonstrated that it is enough for all remaining cases, i.e.

    Proposition [Reduction to Basic cases]:
    If $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$ and $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$, then $\Sigma_{n,2d} \subsetneq \mathcal{P}_{n,2d}$ for all $n \geq 3, 2d \geq 4$ and $(n, 2d) \neq (3, 4)$.

    Idea of Proof:

$$f \in \mathcal{P}_{n,2d} \setminus \Sigma_{n,2d} \Rightarrow \begin{cases} f \in \mathcal{P}_{n+j,2d} \setminus \Sigma_{n+j,2d} \ \forall\, j \geq 0, \\ \end{cases}$$

1. Hilbert's 1888 Theorem

- Let $\mathcal{P}_{n,2d}$, $\Sigma_{n,2d}$ be the set of n-ary $2d$-ic forms that are **positive semidefinite (psd)**, respectively **sum of squares (sos)**.
- **Theorem (Hilbert, 1888):** $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ if and only if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.
  - The arguments for $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ for $n = 2$ and $2d = 2$ were already known in the late 19th century.

    Hilbert proved $\mathcal{P}_{3,4} = \Sigma_{3,4}$.
  - Conversely, Hilbert proved that $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$ and $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$, and demonstrated that it is enough for all remaining cases, i.e.

    Proposition [Reduction to Basic cases]:
    If $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$ and $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$, then $\Sigma_{n,2d} \subsetneq \mathcal{P}_{n,2d}$ for all $n \geq 3, 2d \geq 4$ and $(n, 2d) \neq (3, 4)$.

    Idea of Proof:

$$f \in \mathcal{P}_{n,2d} \setminus \Sigma_{n,2d} \Rightarrow \begin{cases} f \in \mathcal{P}_{n+j,2d} \setminus \Sigma_{n+j,2d} \ \forall\ j \geq 0, \text{and} \\ x_1^{2i} f \in \mathcal{P}_{n,\ 2d+2i} \setminus \Sigma_{n,\ 2d+2i} \ \forall\ i \geq 0. \end{cases} \qquad \square$$

1. Hilbert's 1888 Theorem

- Examples of psd not sos quaternary quartics and ternary sextics:

1. Hilbert's 1888 Theorem

- Examples of psd not sos quaternary quartics and ternary sextics:
  - **(Motzkin, 1967)**
    $M(x, y, z) := z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2 \in \mathcal{P}_{3,6} \setminus \Sigma_{3,6}$

1. Hilbert's 1888 Theorem

- Examples of psd not sos quaternary quartics and ternary sextics:
  - **(Motzkin, 1967)**

    $M(x,y,z) := z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2 \in \mathcal{P}_{3,6} \setminus \Sigma_{3,6}$
  - **(Robinson, 1969)**

    $$R(x,y,z) := x^6 + y^6 + z^6 - (x^4y^2 + y^4z^2 + z^4x^2 + x^2y^4 + y^2z^4 + z^2x^4) + 3x^2y^2z^2 \in \mathcal{P}_{3,6} \setminus \Sigma_{3,6},$$

    $$W(x,y,z,w) := x^2(x-w)^2 + \big(y(y-w) - z(z-w)\big)^2 + 2yz(x+y-w)(x+z-w) \in \mathcal{P}_{4,4} \setminus \Sigma_{4,4}$$

1. Hilbert's 1888 Theorem

- Examples of psd not sos quaternary quartics and ternary sextics:
  - **(Motzkin, 1967)**

    $M(x, y, z) := z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2 \in \mathcal{P}_{3,6} \setminus \Sigma_{3,6}$
  - **(Robinson, 1969)**

    $$R(x, y, z) := x^6 + y^6 + z^6 - (x^4y^2 + y^4z^2 + z^4x^2 + x^2y^4 + y^2z^4 + z^2x^4) + 3x^2y^2z^2 \in \mathcal{P}_{3,6} \setminus \Sigma_{3,6},$$

    $$W(x, y, z, w) := x^2(x - w)^2 + \big(y(y - w) - z(z - w)\big)^2 + 2yz(x + y - w)(x + z - w) \in \mathcal{P}_{4,4} \setminus \Sigma_{4,4}$$
  - **(Choi and Lam, 1976)**

    $Q(x, y, z, w) := w^4 + x^2y^2 + y^2z^2 + z^2x^2 - 4xyzw \in \mathcal{P}_{4,4} \setminus \Sigma_{4,4},$

    $S(x, y, z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2 \in \mathcal{P}_{3,6} \setminus \Sigma_{3,6}$

2. Even Symmetric forms

- A n-ary $2d$-ic form f is called
  - **symmetric** if $\forall\ \sigma \in S_n$: $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$.

2. Even Symmetric forms

- A n-ary $2d$-ic form f is called
  - **symmetric** if $\forall\ \sigma \in S_n$: $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$.
  - **even symmetric** if it is symmetric and in each term of f every variable has even degree.

2. Even Symmetric forms

- A n-ary $2d$-ic form f is called
  - **symmetric** if $\forall\ \sigma \in S_n$: $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$.
  - **even symmetric** if it is symmetric and in each term of f every variable has even degree.
- $\mathcal{SP}^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f \text{ is even symmetric and psd}\}$

2. Even Symmetric forms

- A n-ary $2d$-ic form f is called
  - **symmetric** if $\forall\ \sigma \in S_n$: $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$.
  - **even symmetric** if it is symmetric and in each term of f every variable has even degree.
- $S\mathcal{P}^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f \text{ is even symmetric and psd}\}$
- $S\Sigma^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f \text{ is even symmetric and sos}\}$

2. Even Symmetric forms

- A n-ary $2d$-ic form f is called
  - **symmetric** if $\forall\ \sigma \in S_n$: $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$.
  - **even symmetric** if it is symmetric and in each term of f every variable has even degree.
- $S\mathcal{P}^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f \text{ is even symmetric and psd}\}$
- $S\Sigma^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f \text{ is even symmetric and sos}\}$
- $\mathcal{Q}(S^e)$: For what pairs $(n, 2d)$ will $S\mathcal{P}^e_{n,2d} \subseteq S\Sigma^e_{n,2d}$?

2. Even Symmetric forms

- A n-ary $2d$-ic form f is called
  - **symmetric** if $\forall\ \sigma \in S_n$: $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$.
  - **even symmetric** if it is symmetric and in each term of f every variable has even degree.
- $S\mathcal{P}^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f \text{ is even symmetric and psd}\}$
- $S\Sigma^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f \text{ is even symmetric and sos}\}$
- $\mathcal{Q}(S^e)$: For what pairs $(n, 2d)$ will $S\mathcal{P}^e_{n,2d} \subseteq S\Sigma^e_{n,2d}$?
- Known:
  - $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if $\underbrace{n = 2, d = 1, (n, 2d) = (3, 4)}_{\text{(by Hilbert's Theorem)}}$

2. Even Symmetric forms

- A n-ary $2d$-ic form f is called
  - **symmetric** if $\forall\ \sigma \in S_n$: $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$.
  - **even symmetric** if it is symmetric and in each term of f every variable has even degree.
- $S\mathcal{P}^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f \text{ is even symmetric and psd}\}$
- $S\Sigma^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f \text{ is even symmetric and sos}\}$
- $\mathcal{Q}(S^e)$: For what pairs $(n, 2d)$ will $S\mathcal{P}^e_{n,2d} \subseteq S\Sigma^e_{n,2d}$?
- Known:
  - $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if $\underbrace{n = 2, d = 1, (n, 2d) = (3, 4)}_{\text{(by Hilbert's Theorem)}}, \underbrace{(n, 4)_{n \geq 4}}_{\text{(C-L-R)}}$

2. Even Symmetric forms

- A n-ary $2d$-ic form f is called
  - **symmetric** if $\forall\ \sigma \in S_n$: $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$.
  - **even symmetric** if it is symmetric and in each term of f every variable has even degree.
- $S\mathcal{P}^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f$ is even symmetric and psd$\}$
- $S\Sigma^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f$ is even symmetric and sos$\}$
- $\mathcal{Q}(S^e)$: For what pairs $(n, 2d)$ will $S\mathcal{P}^e_{n,2d} \subseteq S\Sigma^e_{n,2d}$?
- Known:
  - $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if $\underbrace{n = 2, d = 1, (n, 2d) = (3, 4)}_{\text{(by Hilbert's Theorem)}}, \underbrace{(n, 4)_{n \geq 4}}_{\text{(C-L-R)}}, \underbrace{(3, 8)}_{\text{(Harris)}}$

2. Even Symmetric forms

- A n-ary $2d$-ic form f is called
  - **symmetric** if $\forall\ \sigma \in S_n$: $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$.
  - **even symmetric** if it is symmetric and in each term of f every variable has even degree.
- $S\mathcal{P}^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f$ is even symmetric and psd$\}$
- $S\Sigma^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f$ is even symmetric and sos$\}$
- $\mathcal{Q}(S^e)$: For what pairs $(n, 2d)$ will $S\mathcal{P}^e_{n,2d} \subseteq S\Sigma^e_{n,2d}$?
- Known:
  - $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if $\underbrace{n = 2, d = 1, (n, 2d) = (3, 4)}_{\text{(by Hilbert's Theorem)}}$, $\underbrace{(n, 4)_{n \geq 4}}_{\text{(C-L-R)}}$, $\underbrace{(3, 8)}_{\text{(Harris)}}$
  - $S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d}$ for $(n, 2d) = \underbrace{(n, 6)_{n \geq 3}}_{\text{(C-L-R)}}$

2. Even Symmetric forms

- A n-ary $2d$-ic form f is called
  - **symmetric** if $\forall\ \sigma \in S_n$: $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$.
  - **even symmetric** if it is symmetric and in each term of f every variable has even degree.
- $S\mathcal{P}^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f \text{ is even symmetric and psd}\}$
- $S\Sigma^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f \text{ is even symmetric and sos}\}$
- $\mathcal{Q}(S^e)$: For what pairs $(n, 2d)$ will $S\mathcal{P}^e_{n,2d} \subseteq S\Sigma^e_{n,2d}$?
- Known:
  - $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if $\underbrace{n = 2, d = 1, (n, 2d) = (3, 4)}_{\text{(by Hilbert's Theorem)}}, \underbrace{(n, 4)_{n \geq 4}}_{\text{(C-L-R)}}, \underbrace{(3, 8)}_{\text{(Harris)}}$
  - $S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d}$ for $(n, 2d) = \underbrace{(n, 6)_{n \geq 3}}_{\text{(C-L-R)}}, (3, 10)$

2. Even Symmetric forms

- A n-ary $2d$-ic form f is called
  - **symmetric** if $\forall\ \sigma \in S_n$: $f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = f(x_1, \dots, x_n)$.
  - **even symmetric** if it is symmetric and in each term of f every variable has even degree.
- $S\mathcal{P}^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f \text{ is even symmetric and psd}\}$
- $S\Sigma^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f \text{ is even symmetric and sos}\}$
- $\mathcal{Q}(S^e)$: For what pairs $(n, 2d)$ will $S\mathcal{P}^e_{n,2d} \subseteq S\Sigma^e_{n,2d}$?
- Known:
  - $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if $\underbrace{n = 2, d = 1, (n, 2d) = (3, 4)}_{\text{(by Hilbert's Theorem)}}, \underbrace{(n, 4)_{n \geq 4}}_{\text{(C-L-R)}}, \underbrace{(3, 8)}_{\text{(Harris)}}$
  - $S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d}$ for $(n, 2d) = \underbrace{(n, 6)_{n \geq 3}}_{\text{(C-L-R)}}, \underbrace{(3, 10), (4, 8)}_{\text{(Harris)}}$.

2. Even Symmetric forms

- i.e. Known answer to $\mathcal{Q}(S^e)$:

deg \ var	2	3	4	5	...
2	✓	✓	✓	✓	...
4	✓	✓	✓	✓	...
6	✓	×	×	×	...
8	✓	✓	×	?	?
10	✓	×	?	?	?
12	✓	?	?	?	?
⋮	⋮	?	?	?	?

where, a tick (✓) denotes a positive answer to $\mathcal{Q}(S^e)$, a cross ($\times$) denotes a negative answer to $\mathcal{Q}(S^e)$, and a (?) denotes an unknown answer to $\mathcal{Q}(S^e)$.

2. Even Symmetric forms

- To get a complete answer to $\mathcal{Q}(\mathcal{S}^e)$, look at:

deg \ var	2	3	4	5	...
2	✓	✓	✓	✓	...
4	✓	✓	✓	✓	...
6	✓	×	×	×	...
8	✓	✓	×	?	?
10	✓	×	?	?	?
12	✓	?	?	?	?
14	✓	?	?	?	?
⋮	⋮	⋮	?	?	?

$(3, 2d)_{d \geq 6}$

2. Even Symmetric forms

- To get a complete answer to $\mathcal{Q}(S^e)$, look at:

deg \ var	2	3	4	5	6	...
2	✓	✓	✓	✓	✓	...
4	✓	✓	✓	✓	✓	...
6	✓	×	×	×	×	...
8	✓	✓	×	?	?	...
10	✓	×	?	?	?	?
12	✓	?	?	?	?	?
14	✓	?	?	?	?	?
⋮	⋮	⋮	?	?	?	?

$(n, 8)_{n \geq 5}$

$(3, 2d)_{d \geq 6}$

2. Even Symmetric forms

- To get a complete answer to $\mathcal{Q}(S^e)$, look at:

deg \ var	2	3	4	5	6	...
2	✓	✓	✓	✓	✓	...
4	✓	✓	✓	✓	✓	...
6	✓	×	×	×	×	...
8	✓	✓	×	?	?	...
10	✓	×	?	?	?	...
12	✓	?	?	?	?	...
14	✓	?	?	?	?	...
⋮	⋮	⋮	⋮	⋮	⋮	⋱

$(n, 8)_{n\geq 5}$

$(3, 2d)_{d\geq 6}$

$(n, 2d)_{n\geq 4, d\geq 5}$

2.1. Analogue of Hilbert's 1888 Theorem for Even Symmetric forms

- **Theorem (G., Kuhlmann, Reznick):** $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (n, 3)_{n \geq 4}$ or $(3, 8)$.

2.1. Analogue of Hilbert's 1888 Theorem for Even Symmetric forms

- **Theorem (G., Kuhlmann, Reznick):** $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (n, 3)_{n \geq 4}$ or $(3, 8)$.
- i.e.

deg \ var	2	3	4	5	6	...
2	✓	✓	✓	✓	✓	...
4	✓	✓	✓	✓	✓	...
6	✓	×	×	×	×	...
8	✓	✓	×	×	×	...
10	✓	×	×	×	×	...
12	✓	×	×	×	×	...
14	✓	×	×	×	×	...
⋮	⋮	⋮	⋮	⋮	⋮	⋱

2.1. Analogue of Hilbert's 1888 Theorem for Even Symmetric forms

- **Theorem (G., Kuhlmann, Reznick):** $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (n, 3)_{n \geq 4}$ or $(3, 8)$.
- **Proof:**
  - give a "Reduction to Basic Cases: $(n, 8)_{n \geq 5}, (n, 2d)_{n \geq 4, d=5,6}$"

2.1. Analogue of Hilbert's 1888 Theorem for Even Symmetric forms

- **Theorem (G., Kuhlmann, Reznick):** $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (n, 3)_{n\geq 4}$ or $(3, 8)$.
- **Proof:**
  - give a "Reduction to Basic Cases: $(n, 8)_{n\geq 5}, (n, 2d)_{n\geq 4, d=5,6}$" by
    - finding appropriate indefinite irreducible even symmetric forms

2.1. Analogue of Hilbert's 1888 Theorem for Even Symmetric forms

- **Theorem (G., Kuhlmann, Reznick):** $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (n, 3)_{n \geq 4}$ or $(3, 8)$.
- **Proof:**
  - give a "Reduction to Basic Cases: $(n, 8)_{n \geq 5}, (n, 2d)_{n \geq 4, d=5,6}$" by
    - finding appropriate indefinite irreducible even symmetric forms,
    - giving a Degree jumping principle

2.1. Analogue of Hilbert's 1888 Theorem for Even Symmetric forms

- **Theorem (G., Kuhlmann, Reznick):** $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (n, 3)_{n \geq 4}$ or $(3, 8)$.
- **Proof:**
  - give a "Reduction to Basic Cases: $(n, 8)_{n \geq 5}, (n, 2d)_{n \geq 4, d=5,6}$" by
    - finding appropriate indefinite irreducible even symmetric forms,
    - giving a Degree jumping principle

Hilbert's reduction:
(4,4),(3,6)

Degree jump:
Multiply by (x_1)^2i

2.1. Analogue of Hilbert's 1888 Theorem for Even Symmetric forms

- **Theorem (G., Kuhlmann, Reznick):** $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (n, 3)_{n\geq 4}$ or $(3, 8)$.
- **Proof:**
  - give a "Reduction to Basic Cases: $(n, 8)_{n\geq 5}, (n, 2d)_{n\geq 4, d=5,6}$" by
    - finding appropriate indefinite irreducible even symmetric forms,
    - giving a Degree jumping principle to find psd not sos even symmetric $n-$ary forms of degree $\begin{cases} 2d + 4r \text{ (for integer } r \geq 2), \text{ and} \\ 2d + 2n \end{cases}$ from given psd not sos even symmetric $n-$ary $2d-$ic forms.

2.1. Analogue of Hilbert's 1888 Theorem for Even Symmetric forms

- **Theorem (G., Kuhlmann, Reznick):** $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (n, 3)_{n\geq 4}$ or $(3, 8)$.
- **Proof:**
  - give a "Reduction to Basic Cases: $(n, 8)_{n\geq 5}, (n, 2d)_{n\geq 4, d=5,6}$" by
    - finding appropriate indefinite irreducible even symmetric forms,
    - giving a Degree jumping principle to find psd not sos even symmetric $n-$ary forms of degree $\begin{cases} 2d + 4r \text{ (for integer } r \geq 2), \text{ and} \\ 2d + 2n \end{cases}$ from given psd not sos even symmetric $n-$ary $2d-$ic forms.
  - construct explicit forms

2.1. Analogue of Hilbert's 1888 Theorem for Even Symmetric forms

- **Theorem (G., Kuhlmann, Reznick):** $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (n, 3)_{n\geq 4}$ or $(3, 8)$.
- **Proof:**
  - give a "Reduction to Basic Cases: $(n, 8)_{n\geq 5}, (n, 2d)_{n\geq 4, d=5,6}$" by
    - finding appropriate indefinite irreducible even symmetric forms,
    - giving a Degree jumping principle to find psd not sos even symmetric $n-$ary forms of degree $\begin{cases} 2d + 4r \text{ (for integer } r \geq 2), \text{ and} \\ 2d + 2n \end{cases}$ from given psd not sos even symmetric $n-$ary $2d-$ic forms.
  - construct explicit forms
    - $f \in S\mathcal{P}^e_{n,8} \setminus S\Sigma^e_{n,8}$ for $n \geq 5$,

2.1. Analogue of Hilbert's 1888 Theorem for Even Symmetric forms

- **Theorem (G., Kuhlmann, Reznick):** $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (n, 3)_{n\geq 4}$ or $(3, 8)$.
- **Proof:**
  - give a "Reduction to Basic Cases: $(n, 8)_{n\geq 5}, (n, 2d)_{n\geq 4, d=5,6}$" by
    - finding appropriate indefinite irreducible even symmetric forms,
    - giving a Degree jumping principle to find psd not sos even symmetric $n-$ary forms of degree $\begin{cases} 2d + 4r \text{ (for integer } r \geq 2), \text{ and} \\ 2d + 2n \end{cases}$ from given psd not sos even symmetric $n-$ary $2d-$ic forms.
  - construct explicit forms
    - $f \in S\mathcal{P}^e_{n,8} \setminus S\Sigma^e_{n,8}$ for $n \geq 5$,
    - $g \in S\mathcal{P}^e_{n,10} \setminus S\Sigma^e_{n,10}$ for $n \geq 4$,

2.1. Analogue of Hilbert's 1888 Theorem for Even Symmetric forms

- **Theorem (G., Kuhlmann, Reznick):** $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (n, 3)_{n\geq 4}$ or $(3, 8)$.
- **Proof:**
  - give a "Reduction to Basic Cases: $(n, 8)_{n\geq 5}, (n, 2d)_{n\geq 4, d=5,6}$" by
    - finding appropriate indefinite irreducible even symmetric forms,
    - giving a Degree jumping principle to find psd not sos even symmetric $n-$ary forms of degree $\begin{cases} 2d + 4r \text{ (for integer } r \geq 2), \text{ and} \\ 2d + 2n \end{cases}$ from given psd not sos even symmetric $n-$ary $2d-$ic forms.
  - construct explicit forms
    - $f \in S\mathcal{P}^e_{n,8} \setminus S\Sigma^e_{n,8}$ for $n \geq 5$,
    - $g \in S\mathcal{P}^e_{n,10} \setminus S\Sigma^e_{n,10}$ for $n \geq 4$,
    - $h \in S\mathcal{P}^e_{n,12} \setminus S\Sigma^e_{n,12}$ for $n \geq 4$.

2.2. Degree jumping principle

- **Lemma 1 (G. Kuhlmann, Reznick):** For $n \geq 3$, the even symmetric real forms $p_n := 4\sum_{j=1}^{n} x_j^4 - 17\sum_{1\leq i<j\leq n} x_i^2 x_j^2$ and $q_n := \sum_{j=1}^{n} x_j^6 + 3\sum_{1\leq i\neq j\leq n} x_i^4 x_j^2 - 100\sum_{1\leq i<j<k\leq n} x_i^2 x_j^2 x_k^2$ are irreducible over $\mathbb{R}$.

2.2. Degree jumping principle

- **Lemma 1 (G. Kuhlmann, Reznick):** For $n \geq 3$, the even symmetric real forms $p_n := 4\sum_{j=1}^n x_j^4 - 17\sum_{1\leq i<j\leq n} x_i^2x_j^2$ and $q_n := \sum_{j=1}^n x_j^6 + 3\sum_{1\leq i\neq j\leq n} x_i^4x_j^2 - 100\sum_{1\leq i<j<k\leq n} x_i^2x_j^2x_k^2$ are irreducible over $\mathbb{R}$.
- **Lemma 2:** Let $f \in \mathcal{P}_{n,2d} \setminus \Sigma_{n,2d}$ and p an irreducible indefinite form of degree r in $\mathbb{R}[x_1,\ldots,x_n]$. Then $p^2f \in \mathcal{P}_{n,2d+2r} \setminus \Sigma_{n,2d+2r}$.

2.2. Degree jumping principle

- **Lemma 1 (G. Kuhlmann, Reznick):** For $n \geq 3$, the even symmetric real forms $p_n := 4\sum_{j=1}^{n} x_j^4 - 17\sum_{1\leq i<j\leq n} x_i^2 x_j^2$ and $q_n := \sum_{j=1}^{n} x_j^6 + 3\sum_{1\leq i\neq j\leq n} x_i^4 x_j^2 - 100\sum_{1\leq i<j<k\leq n} x_i^2 x_j^2 x_k^2$ are irreducible over $\mathbb{R}$.
- **Lemma 2:** Let $f \in \mathcal{P}_{n,2d} \setminus \Sigma_{n,2d}$ and p an irreducible indefinite form of degree r in $\mathbb{R}[x_1, \ldots, x_n]$. Then $p^2 f \in \mathcal{P}_{n,2d+2r} \setminus \Sigma_{n,2d+2r}$.
- Let $\Delta_{n,2d} := S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$

2.2. Degree jumping principle

- **Lemma 1 (G. Kuhlmann, Reznick):** For $n \geq 3$, the even symmetric real forms $p_n := 4\sum_{j=1}^{n} x_j^4 - 17\sum_{1\leq i<j\leq n} x_i^2 x_j^2$ and $q_n := \sum_{j=1}^{n} x_j^6 + 3\sum_{1\leq i\neq j\leq n} x_i^4 x_j^2 - 100\sum_{1\leq i<j<k\leq n} x_i^2 x_j^2 x_k^2$ are irreducible over $\mathbb{R}$.
- **Lemma 2:** Let $f \in \mathcal{P}_{n,2d} \setminus \Sigma_{n,2d}$ and p an irreducible indefinite form of degree r in $\mathbb{R}[x_1, \ldots, x_n]$. Then $p^2 f \in \mathcal{P}_{n,2d+2r} \setminus \Sigma_{n,2d+2r}$.
- Let $\Delta_{n,2d} := S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$
- **Theorem (G. Kuhlmann, Reznick)** [Degree Jumping Principle]:

  Suppose $f \in \Delta_{n,2d}$ for $n \geq 3$, then

  (i) for any integer $r \geq 2$, the form $p_n^{2a} q_n^{2b} f \in \Delta_{n,2d+4r}$, where $r = 2a + 3b$; $a, b \in \mathbb{Z}_+$,

  (ii) $(x_1 \ldots x_n)^2 f \in \Delta_{n,2d+2n}$.

2.2. Degree jumping principle

- **Lemma 1 (G. Kuhlmann, Reznick):** For $n \geq 3$, the even symmetric real forms $p_n := 4\sum_{j=1}^{n} x_j^4 - 17\sum_{1\leq i<j\leq n} x_i^2 x_j^2$ and $q_n := \sum_{j=1}^{n} x_j^6 + 3\sum_{1\leq i\neq j\leq n} x_i^4 x_j^2 - 100\sum_{1\leq i<j<k\leq n} x_i^2 x_j^2 x_k^2$ are irreducible over $\mathbb{R}$.
- **Lemma 2:** Let $f \in \mathcal{P}_{n,2d} \setminus \Sigma_{n,2d}$ and p an irreducible indefinite form of degree r in $\mathbb{R}[x_1, \ldots, x_n]$. Then $p^2 f \in \mathcal{P}_{n,2d+2r} \setminus \Sigma_{n,2d+2r}$.
- Let $\Delta_{n,2d} := S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$
- **Theorem (G. Kuhlmann, Reznick) [Degree Jumping Principle]:**

  Suppose $f \in \Delta_{n,2d}$ for $n \geq 3$, then

  (i) for any integer $r \geq 2$, the form $p_n^{2a} q_n^{2b} f \in \Delta_{n,2d+4r}$, where $r = 2a + 3b$; $a, b \in \mathbb{Z}_+$,

  (ii) $(x_1 \ldots x_n)^2 f \in \Delta_{n,2d+2n}$.

  Proof follows from above Lemmas. □

- **Proposition (G., Kuhlmann, Reznick) [Reduction to basic cases]**: If $\Delta_{n,2d} \neq \varnothing$ for $(n,8)_{n\geq 4}, (n,10)_{n\geq 3}$ and $(n,12)_{n\geq 3}$, then $\Delta_{n,2d} \neq \varnothing$ for $(n,2d)_{n\geq 3, d\geq 7}$.

- **Proposition (G., Kuhlmann, Reznick) [Reduction to basic cases]**: If $\Delta_{n,2d} \neq \varnothing$ for $(n,8)_{n\geq 4}, (n,10)_{n\geq 3}$ and $(n,12)_{n\geq 3}$, then $\Delta_{n,2d} \neq \varnothing$ for $(n,2d)_{n\geq 3, d\geq 7}$.
- Proof:

- **Proposition (G., Kuhlmann, Reznick) [Reduction to basic cases]**: If $\Delta_{n,2d} \neq \varnothing$ for $(n,8)_{n\geq 4}, (n,10)_{n\geq 3}$ and $(n,12)_{n\geq 3}$, then $\Delta_{n,2d} \neq \varnothing$ for $(n,2d)_{n\geq 3, d\geq 7}$.
- Proof:

deg \ var	2	3
2; 4	✓	✓
6	✓	×
8	✓	✓
10	✓	×
12	✓	
14	✓	
16	✓	
18	✓	
⋮	⋮	

- **Proposition (G., Kuhlmann, Reznick): [Reduction to basic cases]**: If $\Delta_{n,2d} \neq \varnothing$ for $(n,8)_{n\geq 4}, (n,10)_{n\geq 3}$ and $(n,12)_{n\geq 3}$, then $\Delta_{n,2d} \neq \varnothing$ for $(n,2d)_{n\geq 3, d\geq 7}$.
- Proof:

deg \ var	2	3
2; 4	✓	✓
6	✓	×
8	✓	✓
10	✓	×
12	✓	$\times^{(2)}_{6+6}$
14	✓	
16	✓	
18	✓	
⋮	⋮	

- **Proposition (G., Kuhlmann, Reznick) [Reduction to basic cases]**: If $\Delta_{n,2d} \neq \varnothing$ for $(n,8)_{n\geq 4}, (n,10)_{n\geq 3}$ and $(n,12)_{n\geq 3}$, then $\Delta_{n,2d} \neq \varnothing$ for $(n,2d)_{n\geq 3, d\geq 7}$.
- Proof:

deg \ var	2	3
2; 4	✓	✓
6	✓	$\times$
8	✓	✓
10	✓	$\times$
12	✓	$\times^{(2)}_{6+6}$
14	✓	$\times^{(1)}_{6+8}$
16	✓	
18	✓	
⋮	⋮	

- **Proposition (G., Kuhlmann, Reznick) [Reduction to basic cases]**: If $\Delta_{n,2d} \neq \varnothing$ for $(n,8)_{n\geq 4}, (n,10)_{n\geq 3}$ and $(n,12)_{n\geq 3}$, then $\Delta_{n,2d} \neq \varnothing$ for $(n,2d)_{n\geq 3, d\geq 7}$.
- Proof:

deg \ var	2	3
2; 4	✓	✓
6	✓	×
8	✓	✓
10	✓	×
12	✓	$\times^{(2)}_{6+6}$
14	✓	$\times^{(1)}_{6+8}$
16	✓	$\times^{(2)}_{10+6}$
18	✓	$\times^{(2)}_{12+6}$
⋮	⋮	

- **Proposition (G., Kuhlmann, Reznick) [Reduction to basic cases]**: If $\Delta_{n,2d} \neq \varnothing$ for $(n,8)_{n\geq 4}, (n,10)_{n\geq 3}$ and $(n,12)_{n\geq 3}$, then $\Delta_{n,2d} \neq \varnothing$ for $(n,2d)_{n\geq 3, d\geq 7}$.
- Proof:

deg \ var	2	3
2; 4	✓	✓
6	✓	×
8	✓	✓
10	✓	×
12	✓	$\times^{(2)}_{6+6}$
14	✓	$\times^{(1)}_{6+8}$
16	✓	$\times^{(2)}_{10+6}$
18	✓	$\times^{(2)}_{12+6}$
⋮	⋮	$\vdots^{(2)}_{+6}$

- **Proposition (G., Kuhlmann, Reznick) [Reduction to basic cases]**: If $\Delta_{n,2d} \neq \varnothing$ for $(n,8)_{n\geq 4}, (n,10)_{n\geq 3}$ and $(n,12)_{n\geq 3}$, then $\Delta_{n,2d} \neq \varnothing$ for $(n,2d)_{n\geq 3, d\geq 7}$.
- Proof:

deg \ var	2	3	4
2; 4	$\checkmark$	$\checkmark$	$\checkmark$
6	$\checkmark$	$\times$	$\times$
8	$\checkmark$	$\checkmark$	$\times$
10	$\checkmark$	$\times$	
12	$\checkmark$	$\times^{(2)}_{6+6}$	
14	$\checkmark$	$\times^{(1)}_{6+8}$	
16	$\checkmark$	$\times^{(2)}_{10+6}$	
18	$\checkmark$	$\times^{(2)}_{12+6}$	
$\vdots$	$\vdots$	$\vdots^{(2)}_{+6}$	

- **Proposition (G., Kuhlmann, Reznick) [Reduction to basic cases]**: If $\Delta_{n,2d} \neq \varnothing$ for $(n,8)_{n\geq 4}, (n,10)_{n\geq 3}$ and $(n,12)_{n\geq 3}$, then $\Delta_{n,2d} \neq \varnothing$ for $(n,2d)_{n\geq 3, d\geq 7}$.
- Proof:

deg \ var	2	3	4
2; 4	✓	✓	✓
6	✓	×	×
8	✓	✓	×
10	✓	×	
12	✓	$\times^{(2)}_{6+6}$	
14	✓	$\times^{(1)}_{6+8}$	$\times^{(1)}_{6+8}$
16	✓	$\times^{(2)}_{10+6}$	$\times^{(1)}_{8+8}$
18	✓	$\times^{(2)}_{12+6}$	
⋮	⋮	$\vdots^{(2)}_{+6}$	

- **Proposition (G., Kuhlmann, Reznick) [Reduction to basic cases]**: If $\Delta_{n,2d} \neq \emptyset$ for $(n,8)_{n\geq 4}, (n,10)_{n\geq 3}$ and $(n,12)_{n\geq 3}$, then $\Delta_{n,2d} \neq \emptyset$ for $(n,2d)_{n\geq 3, d\geq 7}$.
- Proof:

deg \ var	2	3	4
2; 4	✓	✓	✓
6	✓	×	×
8	✓	✓	×
10	✓	×	
12	✓	$\times^{(2)}_{6+6}$	
14	✓	$\times^{(1)}_{6+8}$	$\times^{(1)}_{6+8}$
16	✓	$\times^{(2)}_{10+6}$	$\times^{(1)}_{8+8}$
18	✓	$\times^{(2)}_{12+6}$	$\times^{(1)}_{6+12}$
⋮	⋮	$\vdots^{(2)}_{+6}$	$\vdots^{(1)}_{+4r}$

- **Proposition (G., Kuhlmann, Reznick) [Reduction to basic cases]**: If $\Delta_{n,2d} \neq \varnothing$ for $(n,8)_{n\geq 4}, (n,10)_{n\geq 3}$ and $(n,12)_{n\geq 3}$, then $\Delta_{n,2d} \neq \varnothing$ for $(n,2d)_{n\geq 3, d\geq 7}$.
- Proof:

deg \ var	2	3	4	5	...
2; 4	✓	✓	✓	✓	...
6	✓	×	×	×	...
8	✓	✓	×	×	...
10	✓	×			
12	✓	$\times^{(2)}_{6+6}$			
14	✓	$\times^{(1)}_{6+8}$	$\times^{(1)}_{6+8}$	$\times^{(1)}_{6+8}$	...
16	✓	$\times^{(2)}_{10+6}$	$\times^{(1)}_{8+8}$	$\times^{(1)}_{8+8}$	...
18	✓	$\times^{(2)}_{12+6}$	$\times^{(1)}_{6+12}$	$\times^{(1)}_{6+12}$	...
$\vdots$	$\vdots$	$\vdots^{(2)}_{+6}$	$\vdots^{(1)}_{+4r}$	$\vdots^{(1)}_{+4r}$	$\ddots$

- **Proposition (G., Kuhlmann, Reznick) [Reduction to basic cases]**: If $\Delta_{n,2d} \neq \emptyset$ for $(n,8)_{n\geq 4}, (n,10)_{n\geq 3}$ and $(n,12)_{n\geq 3}$, then $\Delta_{n,2d} \neq \emptyset$ for $(n,2d)_{n\geq 3, d\geq 7}$.
- Proof:

deg \ var	2	3	4	5	...
2; 4	✓	✓	✓	✓	...
6	✓	×	×	×	...
8	✓	✓	×	×	...
10	✓	×	×	×	...
12	✓	$\times^{(2)}_{6+6}$	×	×	...
14	✓	$\times^{(1)}_{6+8}$	$\times^{(1)}_{6+8}$	$\times^{(1)}_{6+8}$	...
16	✓	$\times^{(2)}_{10+6}$	$\times^{(1)}_{8+8}$	$\times^{(1)}_{8+8}$	...
18	✓	$\times^{(2)}_{12+6}$	$\times^{(1)}_{6+12}$	$\times^{(1)}_{6+12}$	...
$\vdots$	$\vdots$	$\vdots^{(2)}_{+6}$	$\vdots^{(1)}_{+4r}$	$\vdots^{(1)}_{+4r}$	$\ddots$

□

2.3. Explicit examples completing answer to $\mathcal{Q}(S^e)$

- For $m \geq 2$, let
  $$L_{2m+1} := m(m+1)\sum_{i<j}(x_i - x_j)^4 - \Big(\sum_{i<j}(x_i - x_j)^2\Big)^2$$

2.3. Explicit examples completing answer to $\mathcal{Q}(S^e)$

- For $m \geq 2$, let
  $$L_{2m+1} := m(m+1)\sum_{i<j}(x_i - x_j)^4 - \Big(\sum_{i<j}(x_i - x_j)^2\Big)^2$$
- Let $M_r(x_1, \ldots, x_n) := x_1^r + \ldots + x_n^r$ for an integer $r \geq 1$.

2.3. Explicit examples completing answer to $\mathcal{Q}(S^e)$

- For $m \geq 2$, let
  $$L_{2m+1} := m(m+1)\sum_{i<j}(x_i - x_j)^4 - \Big(\sum_{i<j}(x_i - x_j)^2\Big)^2$$
- Let $M_r(x_1, \ldots, x_n) := x_1^r + \ldots + x_n^r$ for an integer $r \geq 1$.
- **Theorem (G., Kuhlmann, Reznick):**
  1. For $m \geq 2$, $G_{2m+1} := L_{2m+1}(x_1^2, \ldots, x_{2m+1}^2) \in \Delta_{2m+1,8}$,

2.3. Explicit examples completing answer to $\mathcal{Q}(S^e)$

- For $m \geq 2$, let
  $L_{2m+1} := m(m+1)\sum_{i<j}(x_i - x_j)^4 - \Big(\sum_{i<j}(x_i - x_j)^2\Big)^2$
- Let $M_r(x_1, \ldots, x_n) := x_1^r + \ldots + x_n^r$ for an integer $r \geq 1$.
- **Theorem (G., Kuhlmann, Reznick):**
  1. For $m \geq 2$, $G_{2m+1} := L_{2m+1}(x_1^2, \ldots, x_{2m+1}^2) \in \Delta_{2m+1,8}$,
  2. For $m \geq 2$, $D_{2m} := L_{2m+1}(x_1^2, \ldots, x_{2m}^2, 0) \in \Delta_{2m,8}$.

2.3. Explicit examples completing answer to $\mathcal{Q}(S^e)$

- For $m \geq 2$, let
  $$L_{2m+1} := m(m+1)\sum_{i<j}(x_i - x_j)^4 - \Big(\sum_{i<j}(x_i - x_j)^2\Big)^2$$
- Let $M_r(x_1, \ldots, x_n) := x_1^r + \ldots + x_n^r$ for an integer $r \geq 1$.
- **Theorem (G., Kuhlmann, Reznick):**
  1. For $m \geq 2$, $G_{2m+1} := L_{2m+1}(x_1^2, \ldots, x_{2m+1}^2) \in \Delta_{2m+1,8}$,
  2. For $m \geq 2$, $D_{2m} := L_{2m+1}(x_1^2, \ldots, x_{2m}^2, 0) \in \Delta_{2m,8}$.
  3. For $n \geq 4$, $T_n(x_1, \ldots, x_n) := M_2(M_2^3 - 5M_2M_4 + 6M_6) \in \Delta_{n,8}$.

2.3. Explicit examples completing answer to $\mathcal{Q}(S^e)$

- For $m \geq 2$, let
  $L_{2m+1} := m(m+1)\sum_{i<j}(x_i - x_j)^4 - \left(\sum_{i<j}(x_i - x_j)^2\right)^2$
- Let $M_r(x_1, \ldots, x_n) := x_1^r + \ldots + x_n^r$ for an integer $r \geq 1$.
- **Theorem (G., Kuhlmann, Reznick):**
  1. For $m \geq 2$, $G_{2m+1} := L_{2m+1}(x_1^2, \ldots, x_{2m+1}^2) \in \Delta_{2m+1,8}$,
  2. For $m \geq 2$, $D_{2m} := L_{2m+1}(x_1^2, \ldots, x_{2m}^2, 0) \in \Delta_{2m,8}$.
  3. For $n \geq 4$, $T_n(x_1, \ldots, x_n) := M_2(M_2^3 - 5M_2M_4 + 6M_6) \in \Delta_{n,8}$.
  4. For $n \geq 4$, $P_n(x_1, \ldots, x_n) := (nM_4 - M_2^2)(M_2^3 - 5M_2M_4 + 6M_6) \in \Delta_{n,10}$.

2.3. Explicit examples completing answer to $\mathcal{Q}(S^e)$

- For $m \geq 2$, let
  $$L_{2m+1} := m(m+1)\sum_{i<j}(x_i - x_j)^4 - \Big(\sum_{i<j}(x_i - x_j)^2\Big)^2$$
- Let $M_r(x_1, \ldots, x_n) := x_1^r + \ldots + x_n^r$ for an integer $r \geq 1$.
- **Theorem (G., Kuhlmann, Reznick):**
  1. For $m \geq 2$, $G_{2m+1} := L_{2m+1}(x_1^2, \ldots, x_{2m+1}^2) \in \Delta_{2m+1,8}$,
  2. For $m \geq 2$, $D_{2m} := L_{2m+1}(x_1^2, \ldots, x_{2m}^2, 0) \in \Delta_{2m,8}$.
  3. For $n \geq 4$, $T_n(x_1, \ldots, x_n) := M_2(M_2^3 - 5M_2M_4 + 6M_6) \in \Delta_{n,8}$.
  4. For $n \geq 4$, $P_n(x_1, \ldots, x_n) := (nM_4 - M_2^2)(M_2^3 - 5M_2M_4 + 6M_6)$ $\in \Delta_{n,10}$.
  5. For $n \geq 3$,
     $R_n(x_1, \ldots, x_n) := (M_2^3 - 3M_2M_4 + 2M_6)(M_2^3 - 5M_2M_4 + 6M_6)$ $\in \Delta_{n,12}$. $\square$

References

M.D. Choi, T.Y. Lam, An old question of Hilbert, Proc. Conf. quadratic forms, Kingston 1976, Queen's Pap. Pure Appl. Math.46 (1977), 385-405.

M.D. Choi, T.Y. Lam and B. Reznick, Even Symmetric Sextics, Math. Z. 195 (1987), 559-580.

C. Goel, S. Kuhlmann, B. Reznick, On the Choi-Lam Analogue of Hilbert's 1888 theorem for symmetric forms, arXiv:1505.08145, 2015.

C. Goel, S. Kuhlmann, B. Reznick, The Analogue of Hilbert's 1888 theorem for even symmetric forms, arXiv:1509.07482, 2015.

W. R. Harris, Real Even Symmetric Ternary Forms, J. Algebra 222 (1999), no. 1, 204-245.

D. Hilbert, Über die Darstellung definiter Formen als Summe von Formenquadraten, Math. Ann., 32 (1888), 342-350; Ges. Abh. 2, 154-161, Springer, Berlin, reprinted by Chelsea, New York, 1981.

Thank you for your attention!