On polynomial, regular and Nash images of Euclidean spaces

(joint work with JM Gamboa & C. Ueno)

SCHEDULE

§1. Statement of the problem

Definitions:

A map $f := (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^m$ is $\left\{\begin{array}{l}\text{polynomial}\\ \text{regular}\\ \text{Nash}\end{array}\right\}$ if each component f_k is $\left\{\begin{array}{l}\text{polynomial}\\ \text{regular}\\ \text{Nash}\end{array}\right.$

- f_k is regular if $f_k = \frac{g_k}{h_k}$ where g_k, h_k are polynomials and $\{h_k = 0\} = \emptyset$
- f_k is Nash if it is smooth and semialgebraic $\longleftrightarrow$ f_k is analytic and there exists a polynomial $P_k \in \mathbb{R}[x][y] \setminus \{0\}$ such that $P_k(x, f_k(x)) = 0 \ \forall x \in \mathbb{R}^m$.

Problems proposed by Gamboa at the 1990 Oberwolfach RAG week:

(1) To characterize the sets $S \subseteq \mathbb{R}^n$ that are the image of a polynomial map $f : \mathbb{R}^m \to \mathbb{R}^n$ (open sets deserve special attention in relation to Jacobian conjecture, see Jelonek 1980)

(2) Same problem for regular maps

(3) Open quadrant problem: To determine if $Q := \{x > 0, y > 0\} \subseteq \mathbb{R}^2$ is a polynomial image of $\mathbb{R}^2$.

Digression: Greeks before going to war consult Delph's oracle. Gamboa consulted Nagoya's Oracle (Shiota) to determine the difficulty of the problems:

Polynomial case: Very very difficult (Translation: Impossible)

Regular case: Very difficult (Tradation: Almost impossible)

He suggested to study the Nash case and Gamboa also announced in the 1990 Oberwolfach RAG Shiota's conjecture concerning Nash images:

"A semialgebraic set $S \subseteq \mathbb{R}^n$ of dimension d is a Nash image of $\mathbb{R}^d$ $\Leftrightarrow$ S is pure dimensional and there exists an analytic path $\alpha: [0,1] \longrightarrow S$ whose image meets all the connected components of the set of regular points * of S"

* Regular points of a semialgebraic set S

(1) $Z \subseteq \mathbb{C}^n$ algebraic set of dimension d. $z \in \mathrm{Reg}(Z) \longleftrightarrow \exists$ analytic diffeo $\varphi: \Delta^d(0,\varepsilon) \longrightarrow W^z \overset{ab}{\subset} Z \Leftrightarrow$ the ring $\mathcal{P}(Z)_{\mathfrak{m}_z}$ is a regular local ring.

(2) $X \subseteq \mathbb{R}^n$ algebraic set. $\mathrm{Reg}(X) = X \cap \mathrm{Reg}(\tilde{X})$: $\tilde{X}$ is a complexification of X

(3) $\mathrm{Reg}(S) := \mathrm{Int}_{\mathrm{Reg}(\overline{S}^{zar})} (S \cap \mathrm{Reg}(\overline{S}^{zar}))$

To simplify notations:

$p(S) := \inf\{p \geq 1 : S = f(\mathbb{R}^p),\ f: \mathbb{R}^p \to \mathbb{R}^n \text{ is polynomial}\}$

$r(S) := \inf\{r \geq 1 : S = f(\mathbb{R}^r),\ f: \mathbb{R}^r \to \mathbb{R}^n \text{ is } \cancel{\text{polynomial}} \text{ regular}\}$

$n(S) := \inf\{m \geq 1 : S = f(\mathbb{R}^m),\ f: \mathbb{R}^m \to \mathbb{R}^n \text{ is Nash}\}$

$\dim(S) \leq n(S) \leq r(S) \leq p(S) \leq +\infty$ (Special interest: When $i(S) = \dim(S)$?

① S is a polynomial image $\Leftrightarrow p(S) < +\infty$.

② S is a regular image $\Leftrightarrow r(S) < +\infty$.

③ S is a Nash image $\Leftrightarrow n(S) < +\infty$.

__Thm__: $n(S) \in \{d, +\infty\}$: $d := \dim(S)$

__Feeling__: $p(S) \in \{d, d+1, +\infty\}$ $\quad d := \dim(S)$
$r(S) \in \{d, d+1, +\infty\}$

__Alternatively__: Is there $S \subseteq \mathbb{R}^n$: $\dim(S) < r(S) < p(S) < +\infty$?

Tarski Seidenberg: $S \subseteq \mathbb{R}^n$ with $p(S), r(S)$ ó $n(S) < +\infty \Rightarrow S$ is a semialgebraic set

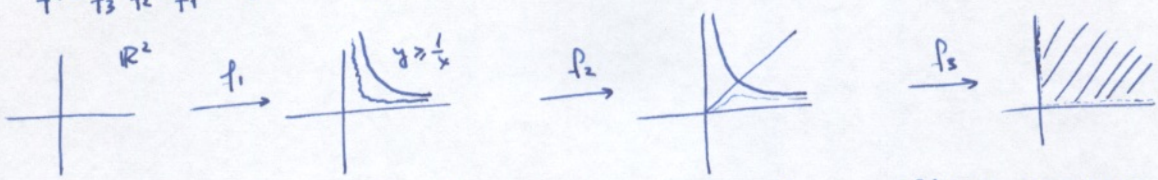
§2. __The open quadrant problem__ (The first demanding result)

Is the open quadrant $Q = \{x > 0, y > 0\}$ a polynomial image of $\mathbb{R}^2$?

1st solution: Presented in the 2002 Oberwolfach RAG week. Required computer assistance for Sturm's algorithm

2^{nd} solution: The shortest proof (ArXiv: 1502.07866)

$f := f_3 \circ f_2 \circ f_1 : \mathbb{R}^2 \to \mathbb{R}^2$



$(x,y) \mapsto ((xy-1)^2+x^2, (xy-1)^2+y^2)$; $(x,y) \mapsto (x, y(xy-2)^2+x(xy-1)^2)$; $(x,y) \mapsto (x(xy-2)^2+\frac{1}{2}xy^2, y)$

3^{rd} solution: The sparsest known polynomial map to represent the open quadrant. A topological argument shows that the image of the following map is $\mathcal{Q}$.

$$f(x,y) := \left((x^2y^4+x^4y^2-y^2-1)^2+x^6y^4,\ (x^6y^2+x^2y^2-x^2-1)^2+x^6y^4 \right)$$

deg 12, 11 monomials; deg 16, 11 monomials

(ArXiv: 1502.08035)

Homework: To find a simpler polynomial map (with respect to degree or number of monomials)

Difficulties: $\begin{cases} \text{To find a polynomial map whose potential image is a prescribed s.a. set } S \subseteq \mathbb{R}^n \\ \text{To prove that the image of the constructed map is } S. \end{cases}$

¿ Are there algorithms to compute explicitly the image of a semialgebraic map?

Strategies to approach the general problem:

(1) Explicit construction of polynomial and regular representations of large families of s.a. sets (so far with piecewise linear boundary)

(2) Search of general obstructions to be polynomial or regular images.

§ 3. The 1-dimensional case (Full characterization)

Let $S \subseteq \mathbb{R}^n$ be a 1-dimensional s.a. set.

3.1. Polynomial case:

$p(S) < +\infty \iff S$ is irreducible ($\mathcal{N}(S)$ is a domain), unbounded, $cl^{zar}_{\mathbb{C}\mathbb{P}^n}(S)$ is an invariant rational curve, $cl^{zar}_{\mathbb{C}\mathbb{P}^n}(S) \cap H_\infty = \{p\}$ and the germ $cl^{zar}_{\mathbb{C}\mathbb{P}^n}(S)_p$ is irreducible

$p(S) < +\infty \Rightarrow p(S) \leq 2$. In addition $p(S) = 1 \iff S$ is closed in $\mathbb{R}^n$

3.2. Regular case:

$r(S) < +\infty \iff S$ is irreducible and $cl^{zar}_{\mathbb{R}\mathbb{P}^n}(S)$ is a rational curve

$r(S) < +\infty \Rightarrow r(S) \leq 2$. In addition $r(S) = 1 \iff \begin{cases} cl_{\mathbb{R}\mathbb{P}^n}(S) = S \text{ or} \\ cl_{\mathbb{R}\mathbb{P}^n}(S) \setminus S = \{p\} \text{ and } \overline{S}^{an}_p \text{ is irreducible} \end{cases}$

3.3. Nash case

$n(S) < +\infty \iff n(S) = 1 \iff S$ is irreducible

§ 4. Piecewise-linear boundary s.a. sets

We have focused on $\left\{ \begin{array}{l} K \text{ convex polyhedron of dim } n \\ \operatorname{Int}(K),\ \mathbb{R}^n \setminus K,\ \mathbb{R}^n \setminus \operatorname{Int}(K) \end{array} \right.$ (Full characterization)

$n \geq 2$

Thm: $r(K) = r(\operatorname{Int}(K)) = n$ if $n \geq 2$

Recession cone: $\vec{C}(K) := \{ \vec{v} \in \mathbb{R}^n : p + \lambda \vec{v} \in K \ \forall \lambda \geq 0 \text{ and } p \in K \}$

$$\vec{C}(K) \neq \{0\} \iff K \text{ is unbounded}$$

Thm: $p(K) < +\infty \Rightarrow \dim \vec{C}(K) = n \Rightarrow p(K) = n \ \& \ p(\operatorname{Int}(K)) \leq n+1$

Thm: $p(\operatorname{Int}(K)) = n \iff \dim \vec{C}(K) = n$ & no bounded faces of dimension $n-1$

$S = \mathbb{R}^n \setminus K$, $\bar{S} = \mathbb{R}^n \setminus \operatorname{Int}(K)$

Thm. $p(S) < +\infty$ or $p(\bar{S}) < +\infty \Rightarrow K \neq \mathbb{R}^{n-1} \times [-a, a] \Rightarrow p(S) = p(\bar{S}) = n$

<table>
<tr><td rowspan="2">K pol. convex
$S = \mathbb{R}^n \setminus K$</td><td colspan="2">$n=1$</td><td colspan="2">$n \geq 2$</td></tr>
<tr><td>K bounded</td><td>K unbounded</td><td>K bounded</td><td>K unbounded</td></tr>
<tr><td>$p(K)$</td><td>$+\infty$</td><td>1</td><td>$+\infty$</td><td>$(n, +\infty)$</td></tr>
<tr><td>$p(\operatorname{Int}(K))$</td><td>$+\infty$</td><td>2</td><td>$+\infty$</td><td>$(n, n+1, +\infty)$</td></tr>
<tr><td>$p(S)$</td><td>$+\infty$</td><td>2</td><td colspan="2" rowspan="8">n</td></tr>
<tr><td>$p(\bar{S})$</td><td>$+\infty$</td><td>1</td></tr>
<tr><td>$r(K)$</td><td>1</td><td>1</td></tr>
<tr><td>$r(\operatorname{Int}(K))$</td><td>2</td><td>2</td></tr>
<tr><td>$r(S)$</td><td>$+\infty$</td><td>2</td></tr>
<tr><td>$r(\bar{S})$</td><td>$+\infty$</td><td>1</td></tr>
</table>

§ 5. General properties

$r(S) < +\infty \Rightarrow S$ is s.a. set, pure dimensional, irreducible, connected by rational paths

$p(S) < +\infty \Rightarrow$ In addition, S is connected by parametric semilines, unbounded or a singleton and with unbounded or singleton projections.

B) Advances properties

Thm: $p(S) < +\infty \Rightarrow S_\infty := cl_{\mathbb{R}\mathbb{P}^n}(S) \cap H_\infty$ is connected (inspired by Jelonek 1999, 2002)

RmK: Not true for $r(S)$.

Questions: (1) Let $S_0 \subseteq H_\infty$ be a connected s.a. set. Is there a polynomial map $f: \mathbb{R}^m \to \mathbb{R}^n$ such that $f(\mathbb{R}^m)_\infty = S_0$? For $n=2$ true

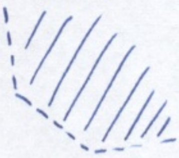
(2) Let $S_0 \subseteq H_\infty$ be a s.a. set. Is there a regular map $f: \mathbb{R}^m \to \mathbb{R}^n$ such that $f(\mathbb{R}^m)_\infty = S_0$? For $n=2$ and S_0 finite true

Thm: If $p(S) = \dim(S) = d$, then:

(1) Let $T := (cl(S) \setminus S)_{d-1}$. $\forall x \in T \ \exists \Gamma$ parametric semiline, $x \in \Gamma \subset \overline{T}^{zar} \cap cl(S)$.

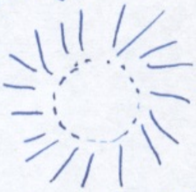
(2) If $d=2$, $T \subset \bigcup_{i=1}^{r} \Gamma_i \subset \overline{T}^{zar} \cap cl(S)$: Γ_i is a parametric semiline.

¿ Are the following s.a. sets polynomial images of $\mathbb{R}^2$?

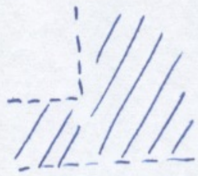


NO, but $p(S) = 3$, $p(\overline{S}) = 2$

NO



No but $p(S) = 3$
$p(\overline{S}) = 2$. In general
$p(\mathbb{R}^n \setminus \overline{B}(0,1)) = n+1$
$p(\mathbb{R}^n \setminus B(0,1)) = n$



Yes $p(S) = 2$
For details email C. Ueno

§6. The Nash case (Full characterization)

Theorem (2015) Let S be a semialgebraic set of dimension d. The following assertions are equivalent:

(1) $n(S) = d$
(2) $n(S) < +\infty$
(3) S is connected by Nash paths
(4) S is connected by analytic paths
(5) S is pure dimensional and there exists a Nash path $\alpha: [0,1] \to S$ such that $\operatorname{im}(\alpha)$ meets all the connected components of $\operatorname{Reg}(S)$
(6) S is pure dimensional and there exists an analytic path $\alpha: [0,1] \to S$ such that $\operatorname{im}(\alpha)$ meets all the connected components of $\operatorname{Reg}(S)$ (Shiota's conjecture)

Proof involves: (1) Resolution of singularities

(2) Extension of Nash manifolds with boundary, Nash double of a Nash manifold with boundary.

(3) Relative approximation of S^{ν} maps on Nash manifolds with boundary

(4) Triangulations of Nash manifolds with Nash strata, etc.

Remarks: • K convex polygon. Then $n(K) = n(\mathrm{Int}(K)) = n(S) = n(\overline{S}) = \dim(S)$

• Irreducible $\not\Rightarrow$ connected by Nash paths
$\Leftarrow$

Particular due case: Let $H \subset \mathbb{R}^n$ be a connected Nash manifold with boundary of dimension d. Then H is a Nash image of $\mathbb{R}^d$.

CONSEQUENCES

(1) Let $S \subseteq \mathbb{R}^n$ be an irreducible pure dimensional s.a. set with $cl(S)$ arc-symmetric*. Then $n(S) = \dim(S)$.

* arc-symmetric (Kurdyka, 1988): if $\alpha: (-1,1) \to \mathbb{R}^n$ analytic arc satisfies $\alpha(-1,0) \subset S$ then $\alpha(-1,1) \subset S$.

(2) Elimination of inequalities (converse of Tarski-Seidenberg)

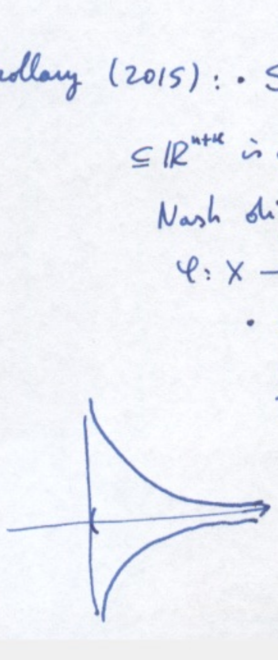
(a) Motzkin (1967): $S \subseteq \mathbb{R}^n$ s.a. is $\pi(X) = S$ for an algebraic set $X \subseteq \mathbb{R}^{n+1}$ (very reducible X and very complicated construction)

(b) Andradas - Gamboa (1996): $S \subseteq \mathbb{R}^n$ closed semialgebraic set $+ \overline{S}^{zar}$ irreducible $\Rightarrow$ $S = \pi(X)$ for an irreducible algebraic set $X \subseteq \mathbb{R}^{n+k}$

(c) Pecker (1990): • Simplify Motzkin construction (simplest and clearer)
• $S \subseteq \mathbb{R}^n$ locally closed s.a. set with an interior point $\Rightarrow$ $S = \pi(X)$ for an irreducible algebraic set $X \subseteq \mathbb{R}^{n+1}$

(d) Corollary (2015): • $S \subseteq \mathbb{R}^n$ s.a. set connected by Nash paths $\Rightarrow$ $S = \pi(X)$ where $X \subseteq \mathbb{R}^{n+k}$ is a non-singular algebraic set whose connected components C_i are Nash diffeomorphic to $\mathbb{R}^d$. Moreover, $\pi(C_i) = X$ and there exist $\varphi: X \to X$ automorphism such that $\varphi(C_i) = C_j$

• For general semialgebraic sets use the decomposition of S into finitely many connected components by Nash paths



X may be not connected

(3) <u>Compact connected smooth manifolds with boundary</u>

- Nash proved (1952): a compact smooth manifold is diffeomorphic to a union of connected components of an smooth algebraic set
- Akbulut-King (1981): a pair (M,N) of a compact smooth manifold and a closed smooth submanifold (of smaller dimension) is diffeomorphic to a pair (X,Y) where X is a smooth algebraic manifold and Y is a smooth algebraic submanifold

<u>Corollary</u> (2015): A connected compact smooth manifold with boundary of dimension d is the smooth image of $\mathbb{R}^d$

Polynomial and Regular Images of $\mathbb{R}^n$

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CIRM 2015

Introduction

A map $f := (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ is *polynomial* if its components f_k are polynomials. Analogously, f is *regular* if its components can be represented as quotients $f_k = \frac{g_k}{h_k}$ of two polynomials g_k, h_k such that h_k never vanishes on $\mathbb{R}^n$. By Tarski-Seidenberg's principle the image of an either polynomial or regular map is a *semialgebraic set*, that is, it has a description by a finite boolean combination of polynomial equalities and inequalities. In 1990 *Oberwolfach reelle algebraische Geometrie* week Gamboa proposed:

Main Problem. Characterize the semialgebraic sets in $\mathbb{R}^m$ which are either polynomial or regular images of some $\mathbb{R}^n$.

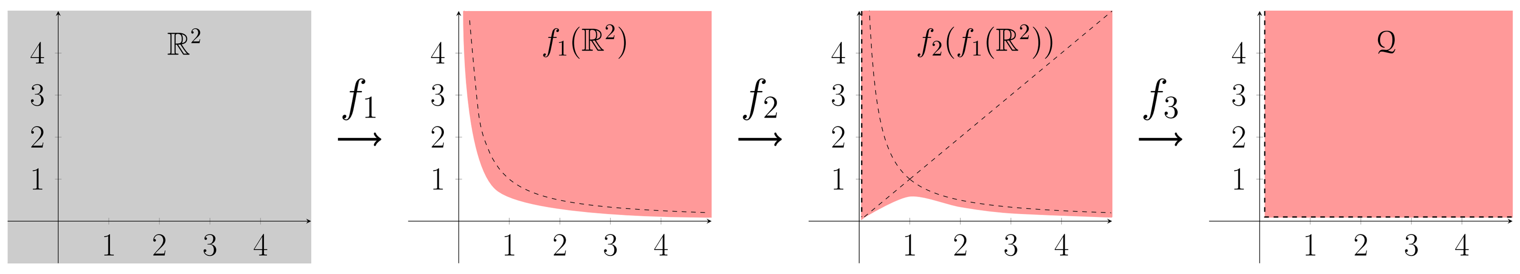
Two approaches to this problem: **(1) Explicit construction** of polynomial and regular representations for large families of semialgebraic sets, so far with **piecewise linear boundary**; and **(2) Search for obstructions** to be polynomial/regular images of $\mathbb{R}^n$. **Potential applications.** Optimization, Positivstellensätze or parametrizations of semialgebraic sets.

The Open Quadrant Problem

Is the set $\mathcal{Q} := \{x > 0, y > 0\} \subset \mathbb{R}^2$ a polynomial image of $\mathbb{R}^2$? Answer: **YES**

First solution. The initial answer was presented in 2002 *Oberwolfach reelle algebraische Geometrie* week. Required computer assistance for Sturm's algorithm.

Second solution. The shortest proof (sketched below).



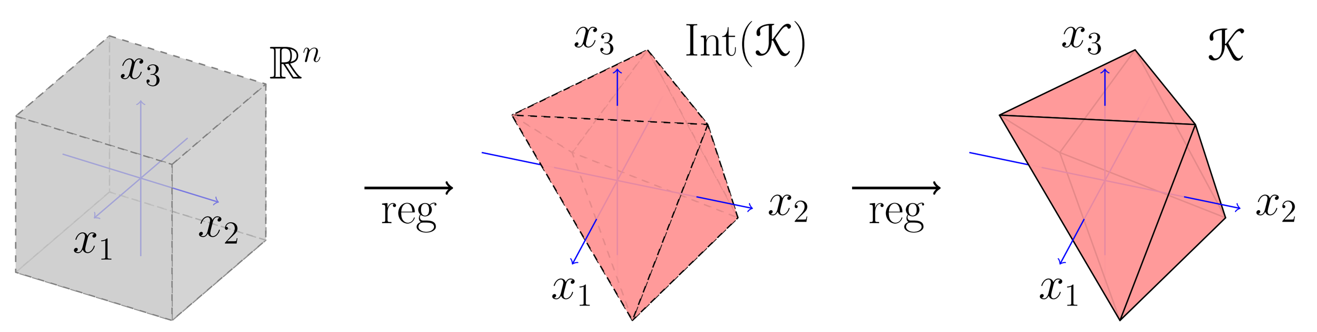
$f_1(x,y) := ((xy-1)^2 + x^2, (xy-1)^2 + y^2)$, $\quad f_2(x,y) := (x, y(xy-2)^2 + x(xy-1)^2)$, $\quad f_3(x,y) := (x(xy-2)^2 + \frac{1}{2}xy^2, y)$.

Third solution. The sparsest (known) polynomial map. A topological argument shows that the image of the map below is $\mathcal{Q}$.

$$f(x,y) := ((x^2y^4 + x^4y^2 - y^2 - 1)^2 + x^6y^4, (x^6y^2 + x^2y^2 - x^2 - 1)^2 + x^6y^4).$$

On Convex Polyhedra

Theorem 1. *An n-dimensional convex polyhedron and its interior are regular images of $\mathbb{R}^n$ $(n \geq 2)$.*



Definition. Let $\mathcal{K} \subset \mathbb{R}^n$ be a convex polyhedron. Its *recession cone* is

$$\vec{\mathcal{C}}(\mathcal{K}) := \{\vec{v} \in \mathbb{R}^n : p + \lambda\vec{v} \in \mathcal{K} \quad \forall p \in \mathcal{K}, \quad \lambda \geq 0\}.$$

Theorem 2. *Let $\mathcal{K} \subsetneq \mathbb{R}^n$ be an unbounded, n-dimensional convex polyhedron whose recession cone $\vec{\mathcal{C}}(\mathcal{K})$ is n-dimensional. Then $\mathcal{K}$ is a polynomial image of $\mathbb{R}^n$. In addition, if $\mathcal{K}$ has not bounded facets, then $\operatorname{Int}(\mathcal{K})$ is also a polynomial image of $\mathbb{R}^n$.*

Theorem 3. *Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional convex polyhedron that is not affinely equivalent to a layer $[-a, a] \times \mathbb{R}^{n-1}$. Then the semialgebraic sets $\mathbb{R}^n \setminus \mathcal{K}$ and $\mathbb{R}^n \setminus \operatorname{Int}(\mathcal{K})$ are polynomial images of $\mathbb{R}^n$.*

Full picture for convex polyhedra

Definition of p **and** r **invariants:**

$\mathrm{p}(\mathcal{S}) := \min\{n \in \mathbb{N} : \mathcal{S} = f(\mathbb{R}^n), f \ \text{ polynomial}\}$

$\mathrm{r}(\mathcal{S}) := \min\{n \in \mathbb{N} : \mathcal{S} = f(\mathbb{R}^n), f \ \text{ regular}\}$

$\mathcal{K}$ conv. pol.	$\mathcal{K}$ bounded		$\mathcal{K}$ unbounded	
$\mathcal{S} = \mathbb{R}^n \setminus \mathcal{K}$	$n = 1$	$n \geq 2$	$n = 1$	$n \geq 2$
$\mathrm{r}(\mathcal{K})$	1	n	1	n
$\mathrm{r}(\operatorname{Int}(\mathcal{K}))$	2		2	
$\mathrm{p}(\mathcal{K})$	$+\infty$		1	$n, +\infty$
$\mathrm{p}(\operatorname{Int}(\mathcal{K}))$			2	$n, n+1, +\infty$
$\mathrm{r}(\mathcal{S})$	$+\infty$	n	2	n
$\mathrm{r}(\overline{\mathcal{S}})$			1	
$\mathrm{p}(\mathcal{S})$			2	
$\mathrm{p}(\overline{\mathcal{S}})$			1	

General Properties

Basic properties. A regular image of $\mathbb{R}^n$ is **connected**, **irreducible** and **pure dimensional**. Polynomial images are in addition either **unbounded** or singletons and have either **unbounded** or singleton **projections**.

Advanced Properties. The *set of points at infinity* of $\mathcal{S} \subset \mathbb{R}^n \subset \mathbb{RP}^n$ is

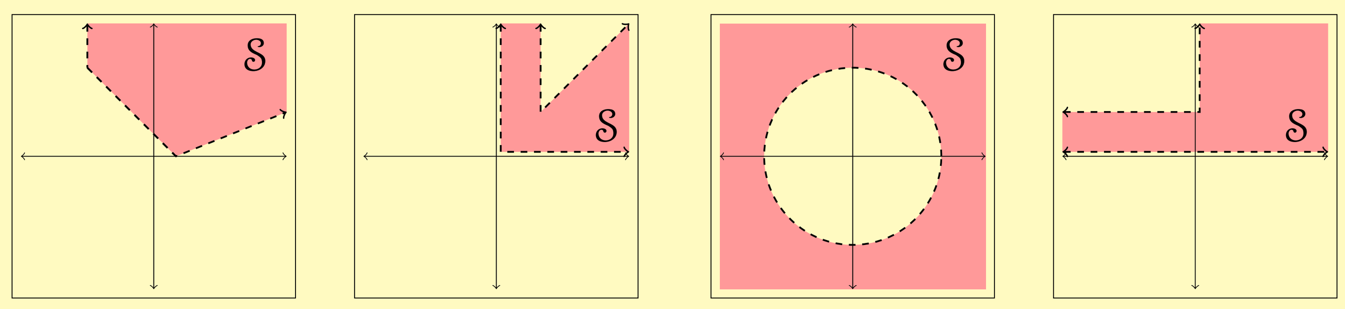
$$\mathcal{S}_\infty := \mathrm{Cl}_{\mathbb{RP}^n}(\mathcal{S}) \cap \mathsf{H}_\infty(\mathbb{R}) \qquad (\mathsf{H}_\infty(\mathbb{R}) \text{ hyperplane at infinity}).$$

Theorem 4. *Let $\mathcal{S} \subset \mathbb{R}^m$ be a polynomial image of $\mathbb{R}^n$. Then $\mathcal{S}_\infty \neq \emptyset$ is connected.*

Remark. This condition does not hold in general for regular images.

Theorem 5. *Let $\mathcal{S} \subset \mathbb{R}^m$ be an n-dimensional polynomial image of $\mathbb{R}^n$. Let $\mathcal{T}$ be the set of points of dimension $n-1$ of $\mathrm{Cl}(\mathcal{S}) \setminus \mathcal{S}$. We have:*
(i) For any $x \in \mathcal{T}$ there is a non-constant polynomial image Γ of $\mathbb{R}$ such that $x \in \Gamma \subset \overline{\mathcal{T}}^{\mathrm{zar}} \cap \mathrm{Cl}(\mathcal{S})$.
(ii) If $n = 2$, $\mathcal{T} \subset \bigcup_{i=1}^r \Gamma_i \subset \overline{\mathcal{T}}^{\mathrm{zar}} \cap \mathrm{Cl}(\mathcal{S})$ where each Γ_i is a polynomial image of $\mathbb{R}$.

Which of the following open sets are polynomial images of $\mathbb{R}^2$?



ANS: The last one

Characterization for the 1-Dimensional Case

Let $\mathcal{S} \subset \mathbb{R}^m$ be a 1-dimensional semialgebraic set.

Theorem 6. *The following assertions are equivalent:*
(i) $\mathcal{S}$ is a polynomial image of $\mathbb{R}^n$ for some $n \geq 1$.
(ii) $\mathcal{S}$ is irreducible, unbounded and $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(\mathcal{S})$ is an invariant rational curve such that $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(\mathcal{S}) \cap \mathsf{H}_\infty(\mathbb{C}) = \{p\}$ and the germ $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(\mathcal{S})_p$ is irreducible.

If that is the case, $\mathrm{p}(\mathcal{S}) \leq 2$. In addition, $\mathrm{p}(\mathcal{S}) = 1 \iff \mathcal{S}$ is closed in $\mathbb{R}^m$.

Theorem 7. *The following assertions are equivalent:*
(i) $\mathcal{S}$ is a regular image of $\mathbb{R}^n$ for some $n \geq 1$.
(ii) $\mathcal{S}$ is irreducible and $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}(\mathcal{S})$ is a rational curve.

If that is the case, then $\mathrm{r}(\mathcal{S}) \leq 2$. In addition, $\mathrm{r}(\mathcal{S}) = 1 \iff$ either $\mathrm{Cl}_{\mathbb{RP}^m}(\mathcal{S}) = \mathcal{S}$, or $\mathrm{Cl}_{\mathbb{RP}^m}(\mathcal{S}) \setminus \mathcal{S} = \{p\}$ and the analytic closure of the germ $\mathcal{S}_p$ is irreducible.

$\mathcal{S}$	$\mathbb{R}$ or $[0, +\infty)$	$\nexists$	$[0, 1)$	$(0, +\infty)$	$(0, 1)$	Any non-rational algebraic curve
$\mathrm{r}(\mathcal{S})$	1	1	1	2	2	$+\infty$
$\mathrm{p}(\mathcal{S})$	1	2	$+\infty$	2	$+\infty$	$+\infty$

Related Problems

A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is *Nash* if each component of f is a *Nash function*, that is, a smooth function with semialgebraic graph. Let $\mathcal{S} \subset \mathbb{R}^m$ be a semialgebraic set of dimension d.

Shiota's conjecture. $\mathcal{S}$ is a Nash image of $\mathbb{R}^d$ if and only if $\mathcal{S}$ is pure dimensional and there exists an analytic path $\alpha : [0,1] \to \mathcal{S}$ whose image meets all connected components of the set of regular points of $\mathcal{S}$.

Corollary 8. *Assume $\mathcal{S}$ is pure dimensional, irreducible and with arc-symmetric closure. Then $\mathcal{S}$ is a Nash image of $\mathbb{R}^d$.*

Corollary 9. *Assume $\mathcal{S}$ is Nash path connected. Then $\mathcal{S}$ is the projection of an irreducible algebraic set $X \subset \mathbb{R}^n$ whose connected components are Nash diffeomorphic to $\mathbb{R}^d$. In addition, each connected component of X maps onto $\mathcal{S}$.*

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