Sums of Squares on Real Projective Varieties

Greg Blekherman
Georgia Tech

Luminy 2015

Sums of Squares on Real Projective Varieties

An exercise in proselytism

Greg Blekherman
Georgia Tech

Luminy 2015

The Question

$X \subset \mathbb{RP}^n$ variety with dense real points.

What is the quantitative relationship between nonnegative polynomials and sums of squares?

quantitative = degree bounds

Important points: X has a canonical presentation: the real radical ideal $I(X)$.

Projective implies no degree cancellation. Rational function certificates are necessary.

Reformulation: How do degree bounds depend on the geometry of X?

Hilbert's Theorem

Theorem: (Hilbert 1888) Nonnegative homogeneous polynomial p is always a sum of squares only in the following three cases:

(1) Bivariate Forms (Univariate Polynomials)

(2) Quadratic Forms

(3) Forms of degree 4 in 3 variables (ternary quartics)

In all other cases there exist nonnegative polynomials that are not sums of squares.

Generalizing HIlbert's Theorem

$R = \mathbb{R}[x_0, \ldots, x_n]/I(X)$ is the coordinate ring of X.

Let $P_X \subset R_2$ be the set of quadratic forms nonnegative on X.

Let $\Sigma_X \subset R_2$ be the set of quadratic forms that are sums of squares of linear forms in R.

Question: Classify real varieties X for which $P_X = \Sigma_X$.

Generalizing HIlbert's Theorem

$R = \mathbb{R}[x_0, \ldots, x_n]/I(X)$ is the coordinate ring of X.

Let $P_X \subset R_2$ be the set of quadratic forms nonnegative on X.

Let $\Sigma_X \subset R_2$ be the set of quadratic forms that are sums of squares of linear forms in R.

Question: Classify real varieties X for which $P_X = \Sigma_X$.

This suffices by using the **Veronese Embedding** ν_d.

Example: $\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5$, $[x : y : z] \to [x^2 : y^2 : z^2, xy, xz, yz]$.
Let $X = \nu_2(\mathbb{P}^2)$. Ternary Quartics: $P_X = \Sigma_X$.

General Theorem

Theorem: (B., G. Smith, M. Velasco) Let $X \subset \mathbb{RP}^n$ be an irreducible nondegenerate projective variety with dense real points.

$P_X = \Sigma_X$ if and only if $X(\mathbb{C})$ is a variety of minimal degree.

Varieties of Minimal Degree

Let $X \subset \mathbb{CP}^n$ be a nondegenerate, irreducible variety. Then

$$\deg X \geq \operatorname{codim} X + 1.$$

If equality is achieved then X is called a *variety of minimal degree*.

Theorem: (Del Pezzo 1886, Bertini 1908) X is a variety of minimal degree if and only if X is one of the following:

(1) Quadratic Hypersurface
(2) Veronese Embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$
(3) Rational Normal Scroll
(4) A (multiple) cone over any of the above

Number of Squares

Theorem: (B., D. Plaumann, R. Sinn, C. Vinzant) Let $X \subset \mathbb{RP}^n$ be a variety of minimal degree. Any polynomial in P_X is a sum of $\dim X + 1$ squares.

Rational Sums of Squares on Curves

Theorem:(B., G. Smith, M. Velasco) Let $X \subset \mathbb{P}^n$ be a real curve of degree d and arithmetic genus p_a. Let $f \in P_{2s}$ be a nonnegative form and let

$$k = \max\left(\operatorname{reg}_{HB} X, \left\lceil \frac{2p_a - 1}{d} \right\rceil\right).$$

Then there exists $h \in \Sigma_{X,2k}$ such that $f \cdot h \in \Sigma_{2s+2k}$.

Remarks:

- $\operatorname{reg}_{HB} X$ is the Hilbert regularity of X.
- Can take $k = d - n + 1 = \deg X - \operatorname{codim} X$.
- The bound depends on simple (complex) geometric invariants of X.
- The degree bound is independent of the degree of f.

Is It Tight?

- Can construct curves in $\mathbb{P}^n$ (any n) where the bound is tight.
- Rational Harnack curves on toric surfaces (and their perturbations).
- Also have examples of curves where the bound is not tight.
- Can lift the bounds from curves to surfaces. For ternary octics can show that $2k = 4$ is the correct bound.

Summer School on Real Algebraic Geometry and Optimization

Organizers: Greg Blekherman, Rainer Sinn, Mauricio Velasco

July 11-15th, 2016
Georgia Tech, Atlanta, Georgia, USA

THANK YOU!