A survey of recent advances in quantitative and algorithmic real algebraic geometry

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Motivations behind quantitative results

- Useful in obtaining upper bounds on numbers of combinatorially distinct configurations – finite sets of points in $\mathbb{R}^d$, or polytopes with fixed number of vertices, oriented matroids etc. (eg. Goodman, Pollack (1986) ...).
- Has become very important in discrete geometry, because of the "polynomial-partitioning" technique introduced by Guth and Katz (2015). The bounds needed here are more refined than the classical ones. (Solymosi and Tao (2013), Zahl (2015), B., Sombra (2015) ... etc.)
- Good quantitative bounds often are indications of the algorithmic complexity of computing the Betti numbers in specific situations. This has in turn formal connections with computational complexity theory in the sense of Blum, Shub and Smale.
- Upper bounds on Betti numbers of a semi-algebraic set translate into lower bounds for the membership in that set in certain models of computations. (Yao (1994), Montana, Morais and Pardo (1996), Gabrielov and Vorobjov (2015)).

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Fixing some notation

- Throughout, R will denote a real closed field.
- Given $P \in \mathrm{R}[X_1, \ldots, X_k]$ we denote by $\mathrm{Z}(P, \mathrm{R}^k)$ the set of zeros of P in R^k.
- Given a finite set $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$, a subset $S \subset \mathrm{R}^k$ is $\mathcal{P}$-semi-algebraic if S is the realization of a Boolean formula with atoms $P = 0$, $P > 0$ or $P < 0$ with $P \in \mathcal{P}$ (we will call such a formula a quantifier-free $\mathcal{P}$-formula).
- We call a semi-algebraic set a $\mathcal{P}$-closed semi-algebraic set if it is defined by a Boolean formula with no negations with atoms $P = 0$, $P \geq 0$, or $P \leq 0$ with $P \in \mathcal{P}$.
- For any semi-algebraic set S, we will denote

$$b(S, \mathbb{F}) = \sum_i b_i(S, \mathbb{F}).$$

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We will usually denote:

- k the dimension of the ambient space.
- $s = \mathrm{card}(\mathcal{P})$.
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Upper bounds on Betti numbers: via effective triangulation

- Upper bounds on the Betti numbers of semi-algebraic sets follow from results on effective triangulation of semi-algebraic sets.
- Effective triangulation in turn uses cylindrical algebraic decomposition – Collins (1976), Wüthrich (1976).
- This yields bounds that are doubly exponential in k. That is,

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Upper bounds on Betti numbers: via the critical point method

- Main idea was to use make a perturbation to reduce to the compact, non-singular, situation and then use Morse theory in order to bound the Betti numbers by the number of critical points of some affine function restricted to the hypersurface. The number of critical point is bounded by Bezout's theorem.
- In this way one obtains (Oleĭnik and Petrovskiĭ (1949), Thom, Milnor (1960s)) $b(\mathrm{Z}(\mathcal{P}, \mathrm{R}^k), \mathbb{F}) \leq d(2d-1)^{k-1}$.
- Generalized to more general semi-algebraic sets – (to $\mathcal{P}$-closed s.a. sets by B.-Pollack-Roy (2005), and then to arbitrary $\mathcal{P}$-s.a. sets Gabrielov-Vorobjov (2005)).
- Generalization uses additional techniques such as generalized Mayer-Vietoris inequalities, homotopic approximations by compact sets (Gabrielov-Vorobjov) etc.

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Upper bounds via critical points (cont).

For completeness ...

Theorem (B.(1999), B.,Pollack,Roy(2005))

Let S be a $\mathcal{P}$-closed semi-algebraic set $S \subset \mathrm{R}^k$, with $s = \mathrm{card}(\mathcal{P})$, and $d = \max_{P \in \mathcal{P}} \deg(P)$, and V a real algebraic variety of dimension $k' \leq k$ also defined by a polynomial of degree at most d. Then,

$$b(S \cap V, \mathbb{F}) \leq \sum_{i=0}^{k'} \sum_{j=0}^{k'-i} \binom{s+1}{j} 6^j d(2d-1)^{k-1} = s^{k'} (O(d))^k.$$

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Upper bounds on Betti numbers: via complex bounds and Smith theory

- Perturbations and then bounding the $\mathbb{Z}_2$-Betti numbers of generic complete intersections in complex projective space using classical formulas for their Euler-Poincaré characteristic (for example, from Hirzebruch's book) and then using Smith inequalities.

- Theorem (Benedetti-Loeser-Risler (1991))

$$b_0(\mathrm{Z}(\mathcal{P},\mathrm{R}^k),\mathbb{Z}_2) \le \left(\frac{1}{2}(\ell+1)k^{\ell-1} + O_\ell(k^{\ell-2})\right) d^k + O_{k,\ell}(d^{k-1}),$$

where $\ell = \mathrm{card}(\mathcal{P})$.

  - Notice that for fixed ℓ, k large enough and $d \to \infty$, the leading coefficient is a polynomial in k (of degree $\ell - 1$), rather being exponential 2^k as in the Oleĭnik-Petrovskiĭ bound, and the leading coefficient of this polynomial is $\frac{1}{2}(\ell+1)$.

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- First exploited by Benedetti, Risler and Loeser (1991) for bounding the number of connected components of real varieties.
- Made into a general method (B. and Rizzie (2015)) for obtaining bounds for $\mathbb{Z}_2$-Betti numbers of real algebraic varieties and semi-algebraic sets, recovering (and improving slightly) all known bounds.
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Upper bounds on Betti numbers : using Kouchnirenko-Bernstein-Khovanskiĭ II

- Two sample theorems.

- Theorem (B., Rizzie (2015))

$$b(Z(\mathcal{P}, \mathrm{R}^k), \mathbb{Z}_2) \le \left(\frac{\ell(3^\ell - 1)}{(\ell - 1)!} k^{\ell-1} + O_\ell(k^{\ell-2}) \right) d^k + O_{k,\ell}(d^{k-1}),$$

where $\ell = \mathrm{card}(\mathcal{P})$.

  - Improves the leading coefficient in the Benedetti-Risler-Loeser bound from $\frac{1}{2}(\ell + 1)$ to $\frac{\ell(3^\ell - 1)}{(\ell - 1)!}$ which goes to 0 as $\ell \to \infty$.
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$$b(\mathrm{Z}(\mathcal{P}, \mathrm{R}^k), \mathbb{Z}_2) \leq \left(\frac{\ell(3^\ell - 1)}{(\ell - 1)!} k^{\ell-1} + O_\ell(k^{\ell-2}) \right) d^k + O_{k,\ell}(d^{k-1}),$$

where $\ell = \mathrm{card}(\mathcal{P})$.

- Improves the leading coefficient in the Benedetti-Risler-Loeser bound from $\frac{1}{2}(\ell+1)$ to $\frac{\ell(3^\ell-1)}{(\ell-1)!}$ which goes to 0 as $\ell \to \infty$.
- Applies to the sum of all the Betti numbers – not just the number of connected components.

Upper bounds on Betti numbers : using Kouchnirenko-Bernstein-Khovanskiĭ II

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Upper bounds on Betti numbers : using Kouchnirenko-Bernstein-Khovanskiĭ III

- Can be used to give “multi-degree” bounds – which are useful in many situations, where different variables can have very different degree dependences.

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  Let $\mathcal{P} \subset \mathrm{R}[\mathbf{X}^{(1)},\ldots,\mathbf{X}^{(p)}]$ where for $1 \le i \le p$, $\mathbf{X}^{(i)} = (X_1^{(i)},\ldots,X_{k_i}^{(i)})$, and $\deg_{\mathbf{X}^{(i)}}(P) \le d_i$, $d_i \ge 2$, for all $P \in \mathcal{P}$. Let $k = \sum_{i=1}^{p} k_i$. Then,

$$b(Z(\mathcal{P},\mathrm{R}^k),\mathbb{Z}_2) \le O(1)^k p^{3k} d_1^{k_1} \cdots d_p^{k_p}.$$

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An application

- The following theorem proved Gabrielov and Vorobjov allows one to bound the Betti numbers of the image of a closed and bounded semi-algebraic set S under a polynomial map $\mathbf{F}$ in terms of the Betti numbers of the iterated fibered product of S over $\mathbf{F}$. More precisely:

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Let $S \subset \mathrm{R}^k$ be a closed and bounded semi-algebraic set, and $\mathbf{F} = (F_1, \ldots, F_m) : \mathrm{R}^k \to \mathrm{R}^m$ be a polynomial map. Then, for for all $p, 0 \leq p \leq m$,

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  Then, for $0 \leq i \leq m$,

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Outline

Upper bounds on the Betti numbers: the quadratic case

- Theorem (Barvinok (1997))
  Let $S \subset \mathrm{R}^k$ be defined by
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- Theorem (Lerario (2012))
  Let $\mathcal{Q} \subset \mathrm{R}[X_0, \ldots, X_k]$ be a set of ℓ quadratic forms. Then,
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  - $S \subset \mathbb{R}^{k_1+k_2}$ a $(\mathcal{P}_1 \cup \mathcal{P}_2)$-closed semi-algebraic set.

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$$b(S, \mathbb{Z}_2) \leq (O(k_2))^{k_1+m+3}(O(sd))^{k_1}, \text{ for } m, k_1 < k_2.$$

Upper bounds on the Betti numbers: the quadratic case III

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Open problems 2

- What about bounds on the Betti numbers of complex varieties defined by polynomials ? Paradoxically, complex methods produce reasonably tight bounds in the real case, but not in the complex case.
- Best bounds in the complex case appear to come from work of Bombieri, Adolphson and Sperber, and Katz – using bounds on exponential sums and descent theory. But these this still do not match in tightness the real bounds.
- Let $V \subset \mathbb{C}^k$ be defined by real polynomials of degrees bounded by d. Let $X \subset V$ be an irreducible component of V. Then is it true that $b(V, \mathbb{Z}_2) \leq O(d)^k$?
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Outline

A real analogue of Bezout inequality I

- (Example in Fulton's book) Let $k = 3$ and let

$$\begin{aligned} Q_1 &= X_3, \\ Q_2 &= X_3, \\ Q_3 &= \sum_{i=1}^{2} \left(\prod_{j=1}^{d} (X_i - j)^2 \right). \end{aligned}$$

The real variety defined by $\{Q_1, Q_2, Q_3\}$ is 0-dimensional, and has d^2 isolated (in R^3) points.

- In particular, this example shows that the (naive version of) Bezout inequality which states that the number of isolated complex zeros of a system of polynomial equations is bounded by the product of the degrees of the polynomials appearing in the system, is not true over if we replace the complex numbers by a real closed field.

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Theorem (B., Barone (2013))

Let

- $Q_1, \ldots, Q_\ell \in \mathrm{R}[X_1, \ldots, X_k]$ *with* $\deg(Q_i) = d_i$;
- *Suppose that*

$$2 \le d_1 \le d_2 \le \frac{1}{k+1} d_3 \le \frac{1}{(k+1)^2} d_4 \le \cdots \le \frac{1}{(k+1)^{\ell-2}} d_\ell.$$

- *For* $1 \le i \le \ell$, *let* $\dim_{\mathrm{R}}(\mathrm{Z}(\{Q_1, \ldots, Q_i\}, \mathrm{R}^k)) \le k_i$ *and let* $k_0 = k$.

Then,

$$b_0(V_\ell, \mathbb{Z}_2) \le O(1)^\ell O(k)^{2k} \left(\prod_{1 \le j < \ell} d_j^{k_{j-1} - k_j} \right) d_\ell^{k_{\ell-1}}.$$

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- Extend the bound to all Betti numbers. A small progress is reported in [B., Rizzie (2015)] where this is proved in the case $\ell = 2$, and $k_1 = k - 1$.
- Improve the dependence on ℓ, k in the bound.

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Fewnomials and sparse systems I

- Theorem (Khovanskiĭ(1980))
  A system of k polynomials in $\mathbb{R}[X_1, \ldots, X_k]$ having $m+k+1$ distinct monomials has at most
  $2^{\binom{m+k}{2}}(k+1)^{m+n}$ *non-degenerate positive solutions.*
  - Consequence of more general theory of real Pfaffian functions.
  - Generalizes Descartes' rule of sign.
  - Using Gale-duality Bihan and Sottile improved this bound (with certain added assumptions) to $O(1)2^{\binom{m}{2}}k^m$.
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  - Key lemma is bounding the number of zeros of a sum of a finite number of analytic functions (in one variable) in terms of the zeros of their Wronskians.
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Outline

Upper bounds on the Betti numbers: the symmetric case I

- For any fixed $d \geq 2$, we have singly exponential lower bound.
- Let $F_{d,k} = \sum_{i=1}^{k} \left(\prod_{j=1}^{d} (X_i - j) \right)^2 - \varepsilon$, and $V_{d,k} = \mathrm{Z}(F_{d,k}, \mathrm{R}\langle\varepsilon\rangle^k)$.
- $b_0(V_{d,k}, \mathbb{F}) = b_{k-1}(V_{d,k}, \mathbb{F}) = d^k$, which is singly exponential in k.
- Notice moreover that each $F_{d,k}$ is a symmetric polynomial.
- Symmetric varieties defined by polynomials of bounded degrees are "simple". For example, for every fixed degree d there is a polynomial-time algorithm to test whether such a variety is empty (Timofte, Riener).
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- Symmetric varieties defined by polynomials of bounded degrees are "simple". For example, for every fixed degree d there is a polynomial-time algorithm to test whether such a variety is empty (Timofte, Riener).
- But clearly from the topological point of view they are not so simple.

Upper bounds on the Betti numbers: symmetric case II

- Theorem (B., Riener (2013))
  Let $P \in \mathrm{R}[X_1,\ldots,X_k]$, be non-negative polynomial of degree bounded by d, and and such that $V = \mathrm{Z}(P,\mathrm{R}^k)$ is invariant under the action of $\mathfrak{S}_k$. Then,

  $$b(V/\mathfrak{S}_k,\mathbb{Q}) \;\; \le \;\; (k)^{2d}(O(d))^{2d+1}.$$

  - Note that $\mathrm{H}^*(V/\mathfrak{S}_k,\mathbb{Q})$ is isomorphic to the isotypic component of $\mathrm{H}^*(V,\mathbb{Q})$ belonging to the trivial representation $\mathbf{1}_{\mathfrak{S}_k}$, and $b(V/\mathfrak{S}_k,\mathbb{Q})$ is its multiplicity.
  - Uses the "degree principle" and equivariant Morse theory.

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More notation

- For any $\mathfrak{S}_k$-symmetric semi-algebraic subset $S \subset \mathrm{R}^k$, and $\lambda \vdash k$, we denote

$$\begin{aligned} m_{i,\lambda}(S, \mathbb{F}) &= \mathrm{mult}(\mathbb{S}^\lambda, \mathrm{H}^i(S, \mathbb{F})), \\ m_\lambda(S, \mathbb{F}) &= \sum_{i \geq 0} m_{i,\lambda}(S, \mathbb{Q}). \end{aligned}$$

Upper bounds on the Betti numbers: the symmetric case III

Theorem (B., Riener (2014))

Let $P \in \mathrm{R}[X_1, \ldots, X_k]$ *be a* $\mathfrak{S}_k$*-symmetric polynomial, with* $\deg(P) \leq d$*. Let* $V = \mathrm{Z}(P, \mathrm{R}^K)$*. Then, for all* $\mu = (\mu_1, \mu_2, \ldots) \vdash k$, $m_\mu(V, \mathbb{Q}) > 0$ *implies that*

$$\mathrm{card}(\{i \mid \mu_i \geq 2d\}) \leq 2d, \mathrm{card}(\{j \mid \tilde{\mu}_j \geq 2d\}) \leq 2d.$$

Moreover,

$$m_\mu(V, \mathbb{F}) \leq k^{O(d^2)} d^d.$$

- Proof uses the degree principle. equivariant Morse theory, equivariant Mayer-Vietoris sequence and some tableau combinatorics. Pieri's rule.

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Open Problems 5

- Conjecture

  For any fixed $d > 0$, there is an algorithm that takes as input the description of a symmetric semi-algebraic set $S \subset \mathrm{R}^k$, defined by a $\mathcal{P}$-closed formula, where $\mathcal{P}$ is a set symmetric polynomials of degrees bounded by d, and computes $m_{i,\lambda}(S, \mathbb{Q})$, for each $\lambda \vdash k$ with $m_{i,\lambda}(S, \mathbb{Q}) > 0$, as well as all the Betti numbers $b_i(S, \mathbb{Q})$, with complexity which is polynomial in $\mathrm{card}(\mathcal{P})$ and k.

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Other quantitative results not discussed in this talk

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- Bounds on the topology of Hausdorff limits.
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- Analogous quantitative results in o-minimal geometry.
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