Signatures of hermitian forms and orderings on algebras with involution

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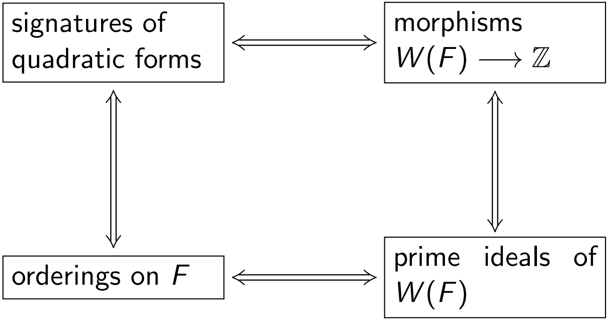
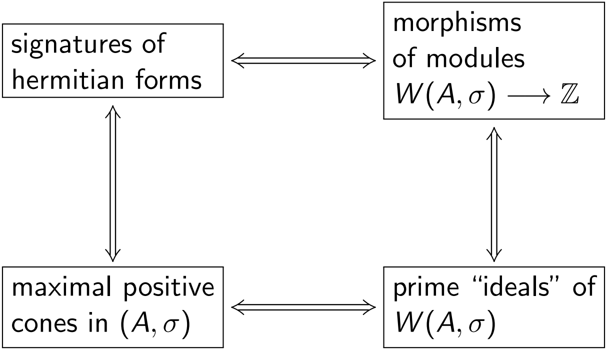
Quadratic forms over a field F, char$F \neq 2$



Hermitian forms over a central simple F-algebra with involution (A, σ)



Setup

- (A, σ) central simple with involution over F:
  A central simple algebra, σ involution on A such that
  $F = Z(A) \cap \mathrm{Sym}(A, \sigma)$
  Note: $[Z(A) : F] \leq 2$

- Main examples:
  - $(M_n(F), {}^t)$
  - $\mathbb{H} = (-1, -1)_{\mathbb{R}}$ with conjugation involution
  - more generally: division algebras with involution

- $W(A, \sigma) :=$ the Witt group of hermitian forms over (A, σ)
  $W(A, \sigma)$ is a $W(F)$-module.

Signatures of hermitian forms

Let $P \in X_F$, F_P a real closure of F at P. Idea: Tensor by F_P.

$$\begin{array}{ccccc}
W(A,\sigma) & \longrightarrow & W(\underbrace{A \otimes_F F_P}_{\cong M_n(D_P)}, \sigma \otimes \mathrm{id}) & \xrightarrow[(\star)]{\cong} & W(M_n(D_P), {}^{-t}) \\
\Big\downarrow \mathrm{sign}_P & & & & \Big\downarrow \cong \\
\mathbb{Z} & & \xleftarrow{\qquad\cong\qquad} & & W(D_P, {}^{-})
\end{array}$$

$$D_P = F_P, F_P(\sqrt{-1}) \text{ or } (-1,-1)_{F_P}$$

Problem: $(\star)$ is not canonical. Can lead to a change of sign.

Solution

$$
\begin{array}{ccccc}
W(A,\sigma) & \longrightarrow & W(\underbrace{A \otimes_F F_P}_{\cong M_n(D_P)}, \sigma \otimes \mathrm{id}) & \xrightarrow[(\star)]{\cong} & W(M_n(D_P), {}^{-t}) \\
\Big\downarrow{\scriptstyle \mathrm{sign}_P} & & & & \Big\downarrow{\scriptstyle \cong} \\
\mathbb{Z} & & \xleftarrow{\qquad\qquad\cong\qquad\qquad} & & W(D_P, {}^{-})
\end{array}
$$

- Define $\mathrm{Nil}[A,\sigma] = \{P \in X_F \mid \mathrm{sign}_P = 0\}$ "Nil orderings"
- **Theorem.** *$\exists \eta \in W(A,\sigma)$ such that $\mathrm{sign}_P\, \eta \neq 0$ for every $P \in X_F \setminus \mathrm{Nil}[A,\sigma]$*
- Take for sign_P^η the one of sign_P, $-\mathrm{sign}_P$ that makes η positive
- In general cannot choose $\eta = \langle 1 \rangle$ as for quadratic forms

Properties of sign^η

(1) Morphism of modules $\mathrm{sign}^\eta_P : W(A,\sigma) \to \mathbb{Z}$

(2) sign^η_P is well-behaved under field extensions, e.g. Knebusch trace formula: for L/F finite and $h \in W(A \otimes_F L, \sigma \otimes \mathrm{id})$,

$$\mathrm{sign}^\eta_P(\mathrm{Tr}^*_{A\otimes_F L} h) = \sum_{P \subseteq Q \in X_L} \mathrm{sign}^{\eta\otimes L}_Q h, \qquad \forall P \in X_F$$

(3) For $h \in W(A,\sigma)$, $\mathrm{sign}^\eta(h) : X_F \to \mathbb{Z}$ is continuous

(4) Pfister's local-global principle and stability index:

$$0 \longrightarrow W_t(A,\sigma) \longrightarrow W(A,\sigma) \xrightarrow{\mathrm{sign}^\eta} C(X_F,\mathbb{Z}) \longrightarrow S(A,\sigma) \longrightarrow 0$$

is exact. The groups $W_t(A,\sigma)$ and $S(A,\sigma)$ are 2-primary torsion groups.

Morphisms of modules

$(f, g) : W(A, \sigma) \to \mathbb{Z}$ is a morphism of modules if

- $f : W(A, \sigma) \to \mathbb{Z}$ is a morphism of groups
- $g : W(F) \to \mathbb{Z}$ is a morphism of rings
- $\forall q \in W(F)\ \forall h \in W(A, \sigma)\ \ f(qh) = g(q)f(h)$

We say that (f, g) and (f', g') are equivalent if there is $a \in \mathbb{Z} \setminus \{0\}$ such that $f = af'$ or $f' = af$.

Theorem.

$$\left\{\text{Signatures } \operatorname{sign}_P^\eta\right\} \xleftrightarrow[\text{canonical}]{\text{one-one}} \left\{\begin{array}{l}\text{equivalence classes of mor-}\\ \text{phisms from } W(A,\sigma) \text{ to } \mathbb{Z}\end{array}\right\}$$

Prime ideals of $W(A, \sigma)$

(I, N) is an ideal of $W(A, \sigma)$ if

- I is an ideal of $W(F)$, N is a submodule of $W(A, \sigma)$
- $I \cdot W(A, \sigma) \subseteq N$

and is a prime ideal if we also have

- $\forall q \in W(F)\ \forall h \in W(A, \sigma) \quad qh \in N \Rightarrow (q \in I \text{ or } h \in N)$

Theorem.

$$\left\{\begin{array}{l}\text{prime ideals of}\\ W(A,\sigma)\end{array}\right\} \xleftrightarrow[\text{canonical}]{\text{one-one}} \left\{\begin{array}{l}\text{equivalence classes of mor-}\\ \text{phisms from } W(A,\sigma) \text{ to } \mathbb{Z}\\ \text{or } \mathbb{Z}/p\mathbb{Z}\end{array}\right\}$$

Positive cones on (A, σ)

$\mathscr{P}$ is a positive cone on (A, σ) if $\mathscr{P} \subseteq \mathrm{Sym}(A, \sigma)$ and

- $\mathscr{P} + \mathscr{P} \subseteq \mathscr{P}$
- $\mathscr{P} \cap -\mathscr{P} = \{0\}$
- $\forall x \in A \quad \sigma(x)\mathscr{P}x \subseteq \mathscr{P}$
- $P(\mathscr{P}) := \{\alpha \in F \mid \alpha\mathscr{P} \subseteq \mathscr{P}\} \in X_F$

Motivation

- $(M_n(F), {}^t)$, and $\mathscr{P} = PSD$ over $P \in X_F$.
- A division algebra and, for $P \in X_F \setminus \mathrm{Nil}[A, \sigma]$:

$$\mathscr{M}_P := \{a \in \mathrm{Sym}(A, \sigma)^\times \mid \mathrm{sign}_P^\eta \langle a \rangle_\sigma \text{ is maximal}\} \cup \{0\}$$

Observations

- $\mathscr{P}$ positive cone iff $-\mathscr{P}$ positive cone
- $\mathscr{P}$ maximal positive cone in $(M_n(\mathbb{R}), {}^t)$ iff $\mathscr{P} = \pm PSD$

Let $\mathscr{P}$ be a maximal positive cone on (A, σ).

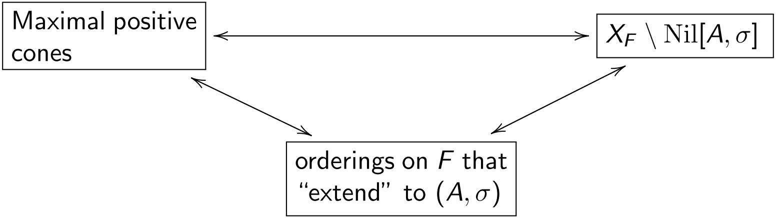
- $\mathscr{P}$ only defines a partial ordering on A
- But, for $a \in \mathrm{Sym}(A, \sigma) \setminus \mathscr{P}$:

$$\sum_{i=1}^{k} u_i \sigma(x_i) a x_i \in -\mathscr{P} \qquad \text{for some } k \in \mathbb{N}, u_i \in P(\mathscr{P}), x_i \in A.$$

Theorem.

$$\left\{ \begin{matrix} \text{maximal } \eta\text{-positive} \\ \text{cones on } (A, \sigma) \end{matrix} \right\} \xleftrightarrow[\text{canonical}]{\text{one-one}} \left\{ \text{non-zero signatures } \mathrm{sign}_P^{\eta} \right\}$$

Intuitively:



Artin-Schreier like results

Theorem.
The following are equivalent

(1) (A,σ) *has a positive cone (is formally real);*

(2) $\exists a \in \mathrm{Sym}(A,\sigma)^\times \quad -a \notin \left\{ \sum_{i=1}^{n} \sigma(x_i) a x_i \,\middle|\, n \in \mathbb{N}, x_i \in A \right\}$.

$X_{(A,\sigma)} := \{\text{maximal positive cones on } (A,\sigma)\}$

Theorem (Procesi-Schacher, simplest case).
Assume that 1 *belongs to every* $\mathscr{P} \in X_{(A,\sigma)}$. *Then*

$$\bigcap_{\mathscr{P} \in X_{(A,\sigma)}} \mathscr{P} = \left\{ \sum_{i=1}^{n} \sigma(x_i) x_i \,\middle|\, n \in \mathbb{N}, x_i \in A \right\}.$$

Topologies on $X_{(A,\sigma)}$

- $X_{(A,\sigma)} \overset{1-1}{\longleftrightarrow} X_F \setminus \mathrm{Nil}[A,\sigma]$
  $\implies X_{(A,\sigma)}$ is equipped with the Harrison topology $\mathcal{T}_H$

- For $a_1, \dots, a_k \in \mathrm{Sym}(A,\sigma)$, the sets

$$H_\sigma(a_1, \dots, a_k) := \{\mathscr{P} \in X_{(A,\sigma)} \mid a_1, \dots, a_k \in \mathscr{P}\}$$

  generate a topology $\mathcal{T}_\sigma$ on $X_{(A,\sigma)}$.

Theorem.

(1) *$\mathcal{T}_\sigma$ is spectral.*

(2) *The patch topology of $\mathcal{T}_\sigma$ is the coarsest topology that makes all global signatures of hermitian forms continuous.*

(3) *The patch topology of $\mathcal{T}_\sigma$ is equal to $\mathcal{T}_H$.*