Globally defined semianalytic sets

Thanks. This is a joint work with F. Broglia and J. F. Fernando. ~~Th~~ Program

1) Motivation
2) Definitions and results
3) The set of points where a real analytic set is not coherent.

1) This work comes directly from H. Cartan, let me explain.

During the fifties of last century the theory of complex analytic spaces was developed mainly in France (Oka - Cartan - Serre - Grothendieck - Sém. Cartan) and in Germany (Rückert, Behnke, Stein, Remmert, Grauert ...). Soon it was remarked that Oka coherence theorem, which is at the base of this development, did not hold true in the real case.

In other words if $X \subset M$ is a real analytic set in a real analytic manifold the ideal sheaf $\mathcal{J}_X \subset \mathcal{O}_M$ of analytic ~~germs~~ function germs vanishing at X needs not to be coherent as in the complex case.

This fact lead to two opposite positions: on one side Grothendieck considered the real case not interesting at all. On the other side Cartan, after finding several wild examples, indicates a smaller class of real analytic sets with more tame behaviour. ②

What does mean wild? For instance

$$X = \{a(z)x^3 - z(x^2+y^2) = 0\} \text{ where } a(z) = \begin{cases} \exp\frac{1}{z^2-1} & \text{for } -1<z<1 \\ 0 & \text{otherwise} \end{cases}$$

The fact that $a(z)$ has an essential singularity at $z = \pm 1$ forces any analytic function $f \in \mathcal{O}(\mathbb{R}^3)$ vanishing on X to be the zero function. There are other examples (see Bruhat-Cartan / Cartan, Whitney-Bruhat) where the notion of irreducible component makes no sense.

The smallest class indicated by Cartan consists of those real analytic sets that lie in a complex Stein space and are the fixed point sets of an anti holomorphic involution. For instance, if $X \subset \mathbb{R}^n$ and there is an invariant Stein neighbourhood Ω of $\mathbb{R}^n$ in $\mathbb{C}^n$, a closed complex ~~[illegible]~~ subset Y invariant subset $Y \subset \Omega$ such that $X = Y \cap \mathbb{R}^n$.

Cartan says that the only ~~rea~~ meaningful notion of real analytic set X in $\mathbb{R}^n$ must refer to a complex analytic set in $\mathbb{C}^n$ giving X by intersection with $\mathbb{R}^n$. He proves that these real analytic sets are the ones that can be defined as the common zero set of finitely many analytic functions on $\mathbb{R}^n$.

He says: "La notion de sous ensemble analytique réel a ainsi un caractère essentiellement global, contrairement à ce qui avait lieu pour les sous-ensembles analytiques complexes"

After Whitney and Bruhat we can define ~~for~~ a real analytic set X in a real analytic manifold M

Def: X is C-analytique if there are $f_1, \dots, f_k \in \mathcal{O}(M)$ such that $X = \{x \in M \mid f_1(x) = \dots = f_k(x) = 0\}$.

Considering also inequalities one comes to the definition of semianalytic set (Łojasiewicz)

Def: $S \subset M$ is semianalytic if for all $x \in M$ there is an open neighbhd U^x such that $S \cap U^x$ is a finite union of sets of the form $\{f = 0, g_1 > 0, \dots, g_s > 0\}$ where $f, g_1, \dots, g_s \in \mathcal{O}(U^x)$. < come per i complex sets

The family of semianalytic sets is not stable by proper analytic maps and this lead to the introduction of subanalytic sets (Łojasiewicz, Hironaka)

→ We are interested in global properties of the ring $\mathcal{O}(M)$. You should consider

~~In our opinion it is a~~ natural ~~question~~ for us to ask whether there is a more global notion of semianalytic set, at least for subsets of a C-analytic set in the sense of Cartan, i.e. whether there exist a class of semianalytic sets, only defined using

analytic functions on the ambient space with (4)
good behaviour with respect to boolean and topo-
logical operation.

A first tentative in this direction was explored by Andradas-Bröcker-Ruiz (with some compactness assumption) and by Andradas-Castilla mainly in dimension 2.

They defined a "global semianalytic set" in a real analytic manifold M as a definable set of the ring $\mathcal{O}(M)$. But for the general case we do not know whether the closure or the connected components of a global semianalytic set is still global.

(No relation with the global subanalytic sets of o-minimal structures. Here nothing is o-minimal)

2) We propose the following definition:

Def $S \subset M$ is a C-semianalytic set if $S = \bigcup_i S_i$ where the union is <u>locally finite</u> and S_i is a global semianalytic basic set $S_i = \{f_i = 0, g_{ij} > 0\}$

This definition is equivalent to the following, more similar to the one of topologies.

Def S is C-semianalytic if and only if $\forall x \in M$ there is an open neighbhd U^x such that $S \cap U^x$ is a global semianalytic subset of M.

Properties

C-semianalytic sets are stable by

- locally finite unions, intersection, complement
- closure and interior part, connected components
- inverse image under analytic maps.
- proper analytic maps between Stein spaces.

More precisely the last property says.

<u>Theorem</u> (direct image theorem) X, Y reduced Stein spaces equipped with anti holom. involutions $\sigma : X \to X$, $\tau : Y \to Y$ such that X^σ and Y^τ are not empty. $f : X \to Y$ proper analytic map.

1) $S \subset X^\sigma$ C-analytic set described by invariant functions on X restricted to X^σ. Then $f(S)$ is <u>C-semianalytic</u> in Y^τ.
2) $E = f^{-1}(Y^\tau) \setminus X^\sigma$, then $f(E \cap S)$ is C-semianalytic
3) If $f^{-1}(Y^\tau) = X^\sigma$ then $f(S)$ is C-semianalytic whenever S is.

Note that a proper map between Stein spaces is finite. We have a nice local caracterisation of such a map, namely: $y \in Y$ $\quad f^{-1}(y) = \{x_1, -, x_p\}$. Put $S = \mathcal{O}(X) \setminus m_{x_1} \cup \cdots \cup m_{x_p}$. Then

$$S^{-1}\mathcal{O}(X) = \mathcal{O}(Y)_{m_y}[H_1 - H_m], \ H_j \in \mathcal{O}(X)$$

Special ~~subsets~~ special subsets of a C-analytic set are known to be semianalytic. ~~Next~~ For a C-analytic set X we define a C-property a property P such that

$$\{x \in X \mid P(x)\} \text{ is a C-semianalytic set}$$

(All those sets are semianalytic: we ask more!)
For instance the following are P-properties

Several sets related to X are known to be semianalytic. Are they C-semianalytic?

- $P_k(x) = \dim_x X = k$

- $N(x) =$ X is not coherent at the point x

All these sets where known to be semianalytic. We only add globality
~~Let us see something about the second one~~

3. The set $N(X)$ where X is not coherent

By Cartan criterion a point $x_0 \in X$ where X is not coherent verifies the following

- Take $\tilde{X}_{x_0}$ the complexification of the germ X_{x_0}. There are points $y \in X$ arbitrarily close to x_0 such that $\hat{X}_y$ is not induced by $\tilde{X}_{x_0}$.

This means that something that was real becomes complex (as real roots disappearing after a double point) and the dimension drops.
Think in Whitney umbrella

$$x^2 - zy^2 \subset \mathbb{R}^3 \qquad x_0 = (0,0,0)$$

or Cartan umbrella $x^3 - z(x^2+y^2) = 0$. (the cone over the cubic)

for $z < 0$

But it is quite possible that the tail lies in some maximal dimension part of X so that $\dim X_{x_0} = \dim X_y$ for y in the tail

$$z(x+y)(x^2+y^2) - x^4 = 0$$

again for $z < 0$ there is a tail embedded in the 2 dimensional part.

Note that, since the tail is included in $\operatorname{Sing} X$ this cannot happen if X is normal, since normalization separates components and the tail is the intersec. of two complex conj. components.

Assume for a moment to have a C-analytic set $X \subset \mathbb{R}^n$ such that all irred. comp. have the same dimension d. Take a ~~a~~ minimal complexification $\tilde{X}$ in $\Omega \subset \mathbb{C}^n$. Then $\dim_{\mathbb{C}} \tilde{X} = d = \dim_{\mathbb{R}} X$

Let Y be a normalization of $\tilde{X}$ with the antiinvolution σ induced by the compl. conj on $\tilde{X}$.

$$\pi : Y \to \tilde{X} \quad \text{is a proper map of Stein spaces}$$

Now the inverse images of tails in X becomes

- tail of Y^σ
- ~~some~~ $E = \pi^{-1}(X) \setminus Y^\sigma$

So the set $N(X)$ where X is not coherent is as follows

$$C_1 = \pi^{-1}(X) \setminus Y^\sigma, \quad C^2 = Y^\sigma \setminus \overline{Y^\sigma \setminus \operatorname{Sing} Y^\sigma}$$

$$A_i = \overline{C_i} \cap \overline{Y^\sigma \setminus \operatorname{Sing} Y^\sigma}$$

then $N(X) = \pi(A_1) \cup \pi(A_2)$. is C-semianalytic by the direct image theorem.

The general case follows the same idea, but it is more complicated. ②

- no need for C-subanalytic.
- our class does not verify

  $\ast$ $\dim S = \dim \overline{S}^{Z}$

~~To get the~~ For those C-semianalytic sets verifying $\ast$ Fernando developed ~~the~~ a theory of irreducible components very similar to the one he did for semialgebraic sets.