Ordered Algebraic Structures and Related Topics
12 -16 October, 2015

Francesca Acquistapace: A class of globally defined semianalytic sets.

We propose a class of semianalytic sets in a C-real analytic set X or in an analytic manifold, which are locally definable by the algebra $\mathcal{O}(X)$ of global analytic functions on X and is stable by topological and boolean operations. We prove that this class is stable by proper analytic maps that admit a proper complexification. We prove that several important subsets of X are in the class, as for instance the set of points where X has dimension k or the set of points where X is not coherent.
This class, in the framework of semianalytic sets, is in some sense, the analog of the class of C-analytic sets defined by Cartan among all real analytic sets.
Work in collaboration with J. Fernando.

Vincent Astier: Signatures of hermitian forms and orderings on some algebras with involution.

Let (A, σ) be an F-algebra with involution where F is a formally real field. We define signatures of hermitian forms over (A, σ) with respect to orderings on F and discuss their properties. These signatures are in canonical correspondence with morphisms of modules from the Witt group $W(A, \sigma)$ into $\mathbb{Z}$ and provide a conceptual framework for the theorem of Procesi and Schacher on sums of hermitian squares in (A, σ), a noncommutative analogue of Artin's solution to Hilbert's 17th problem. This naturally leads to the question of the existence of positive cones on (A, σ). We define such positive cones and discuss some of their properties. This is joint work with Thomas Unger.

Saugata Basu: A survey of recent advances in quantitative and algorithmic real algebraic geometry.

I will survey some of the recent results in quantitative and algorithmic real algebraic focussing on motivations and the new ideas that make these advances possible. In particular, I will discuss better bounds on the number of connected components of realizations of sign conditions, new nearly optimal roadmap algorithms for deciding connectivity of semi-algebraic sets, and new results on the topology of symmetric semi-algebraic sets.

Eberhard Becker: Sums of Powers on Algebraic Function Fields in One Variable.

The talk fits into a general study of sums of n-th powers and the corresponding Pythagoras numbers $P_n(F)$, F an algebraic function field in one variable over $\mathbb{R}$. The results in the quadratic case go back to E. Witt's fundamental work in the 1930's. In particular, $P_2(F) = 2$. About 30 years ago, the present author proved that for all n the number $P_n(F)$ is finite, and a reasonable bound was given. In particular, given the exponent n there exists a number N

such that every sum of n-th powers in F can be represented as a sum of at most N terms of n-th powers.
The talk will sketch two new methods to bound the higher Pythagoras number from above and below.
To obtain better upper bounds in the case of a formally real function field, e.g. $P_4(F) \leq 6$, we first turn to the uniquely determined projective curve over $\mathbb{R}$ and its real locus γ which is a 1-dimensional compact manifold with finitely components. There exists a natural topological representation $F \to C(\gamma, \mathbb{P}^1(\mathbb{R}))$. The main result states that this representation has dense image. The proof invokes differential topology. This result provides deeper insight into the group of units of the real holomorphy ring what leads to sharper upper bounds.
The classical theorem of Riemann-Hurwitz on the behaviour of the genus under finite extensions of function fields can be used to prove the non-existence of roots of equations with coefficients is algebraic function field. The talk will briefly sketch applications to non-real function fields which results in lower bounds for the higher Pythagoras number and the higher stufe.

Anna Blaszczok: Distances of elements in valued field extensions.

Elements of a valued field extension induce cuts in the divisible hull of the value group of the base field. These cuts, called distances, encode important information about the valued field extension. The notion was introduced by F.-V. Kuhlmann and turned out to be meaningful in particular in the study of the structure of defect extensions of valued fields in the positive characteristic case.

We introduce a slight modification of the definition of distance, which carries more information about field extensions. Based on the case of extensions of prime degree, we then describe what kind of information about the field extension is encoded in the distances. We also describe the possible distances of elements in extensions of prime degree. In particular, we consider a classification of defect extensions of prime degree, which is defined by the distances. We also present results which explain the meaning of the classification for problems connected with the structure of defect and immediate extensions.

Greg Blekherman: Degree Bounds in Rational Sums of Squares Representations on Varieties.

Let X be a variety with dense real points. It is well-known that any polynomial p nonnegative on X can be written as a sum of squares of rational functions in the coordinate ring of X. I will present new degree bounds for the rational sums of squares representations in the case X is a curve or a rational surface. The bound can be shown to be tight in many instances and it depends only on the Hilbert series of X.

Ludwig Bröcker: On p-adic semialgebraic sets

We introduce coarse and fine semialgebraic sets over a given p-adic field. Coarse sets are defined by polynomial inequalities in the values and boolean combinations, whereas fine sets are defined by n-th power relations for polynomials. Coarse sets essentially can be written by a single inequality, but only the class of fine sets is stable under projections. We will present some results on separation and description by few polynomial n-th power relations for fine semialgebraic sets.

Quentin Brouette: Stellensatze in closed ordered differential fields

M. Singer introduced the model completion of the theory of ordered differential fields, named by the acronym CODF (closed ordered differential fields). He gave an axiomatisation of that theory. The purpose of this talk is to present analogues of some results of real algebra, in that differential context.

Using the model completeness of CODF and the fact that radical differential ideals are finitely generated, we establish a nullstellensatz and a positivstellensatz for CODF. A consequence of Singer's axioms for CODF is a strong relationship between the derivation and the topology: the density of differential points. Using this density, we obtain another version of the positivstellensatz for CODF and a version of Schmodgen's theorem (originally proved for the real field) for some specific models of CODF.

José F. Fernando: On Polynomial, Regular and Nash Images of Euclidean spaces
Extended abstract. Joint work with Carlos Ueno and J. M. Gamboa.

A map $f := (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ is *polynomial* if its components f_k are polynomials. Analogously, f is *regular* if its components can be represented as quotients $f_k = \frac{g_k}{h_k}$ of two polynomials g_k, h_k such that h_k never vanishes on $\mathbb{R}^n$. By Tarski-Seidenberg's principle the image of an either polynomial or regular map is a *semialgebraic set*, that is, it has a description by a finite boolean combination of polynomial equalities and inequalities. In 1990 *Oberwolfach reelle algebraische Geometrie* week Gamboa proposed a kind of converse problem: *Characterize the semialgebraic sets in $\mathbb{R}^m$ which are either polynomial or regular images of some $\mathbb{R}^n$.*

We have approached this problem in two directions: (1) Explicit construction of polynomial and regular representations for large families of semialgebraic sets, so far with piecewise linear boundary; and (2) Search for obstructions to be polynomial/regular images of $\mathbb{R}^n$.

We schedule next the main obtained results that we will present and explain in our survey.

1. Origin of our approach: The open quadrant problem

Is the set $\mathcal{Q} := \{x > 0, y > 0\} \subset \mathbb{R}^2$ a polynomial image of $\mathbb{R}^2$? Answer: YES.

First solution. The initial answer was presented in 2002 *Reelle algebraische und analytische Geometrie* week. Required computer assistance for Sturm's algorithm.

Second solution. The shortest proof. We have shown that $\mathcal{Q} = (f_3 \circ f_2 \circ f_1)(\mathbb{R}^2)$, where
$f_1(x, y) := ((xy - 1)^2 + x^2, (xy - 1)^2 + y^2)$, $f_2(x, y) := (x, y(xy - 2)^2 + x(xy - 1)^2)$,
$f_3(x, y) := (x(xy - 2)^2 + \frac{1}{2}xy^2, y)$.

Third solution. The sparsest (known) polynomial map. A topological argument shows that $\mathcal{Q}$ is the image of $f(x, y) := ((x^2y^4 + x^4y^2 - y^2 - 1)^2 + x^6y^4, (x^6y^2 + x^2y^2 - x^2 - 1)^2 + x^6y^4)$.

2. On Convex Polyhedra

Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional convex polyhedron:

- *$\mathcal{K}$ and $\operatorname{Int}(\mathcal{K})$ are regular images of $\mathbb{R}^n$ (if $n \geq 2$).*
- *If $\mathcal{K}$ is unbounded and its recession cone $\vec{\mathcal{C}}(\mathcal{K}) := \{\vec{v} \in \mathbb{R}^n : p + \lambda\vec{v} \in \mathcal{K} \quad \forall p \in \mathcal{K}, \quad \lambda \geq 0\}$, then $\mathcal{K}$ is a polynomial image of $\mathbb{R}^n$. In addition, if $\mathcal{K}$ has not bounded facets, then $\operatorname{Int}(\mathcal{K})$ is also a polynomial image of $\mathbb{R}^n$.*
- *If $\mathcal{K}$ is not affinely equivalent to a layer $[-a, a] \times \mathbb{R}^{n-1}$, the semialgebraic sets $\mathbb{R}^n \setminus \mathcal{K}$ and $\mathbb{R}^n \setminus \operatorname{Int}(\mathcal{K})$ are polynomial images of $\mathbb{R}^n$.*

3. General Properties

Basic properties. A regular image of $\mathbb{R}^n$ is connected, irreducible and pure dimensional. Polynomial images are in addition either unbounded or singletons and have either unbounded or singleton projections.

Advanced Properties. The *set of points at infinity* of $\mathcal{S} \subset \mathbb{R}^n \subset \mathbb{RP}^n$ is

$$\mathcal{S}_\infty := \operatorname{Cl}_{\mathbb{RP}^n}(\mathcal{S}) \cap \mathsf{H}_\infty(\mathbb{R}) \qquad (\mathsf{H}_\infty(\mathbb{R}) \text{ hyperplane at infinity}).$$

Points at infinity. Let $\mathcal{S} \subset \mathbb{R}^m$ be a polynomial image of $\mathbb{R}^n$. Then $\mathcal{S}_\infty \neq \emptyset$ is connected. This condition does not hold in general for regular images.

Boundary. (i) $\forall x \in \mathcal{T}$ there exists a non-constant polynomial image Γ of $\mathbb{R}$ such that $x \in \Gamma \subset \overline{\mathcal{T}}^{\text{zar}} \cap \operatorname{Cl}(\mathcal{S})$.

(ii) If $n = 2$, $\mathcal{T} \subset \bigcup_{i=1}^r \Gamma_i \subset \overline{\mathcal{T}}^{\text{zar}} \cap \operatorname{Cl}(\mathcal{S})$ where each Γ_i is a polynomial image of $\mathbb{R}$.

4. Characterization for the one-dimensional case

Let $\mathcal{S} \subset \mathbb{R}^m$ be a 1-dimensional semialgebraic set. Define

$$\operatorname{p}(\mathcal{S}) := \min\{n \in \mathbb{N} : \mathcal{S} = f(\mathbb{R}^n), f \quad \textit{polynomial}\}$$
$$\operatorname{r}(\mathcal{S}) := \min\{n \in \mathbb{N} : \mathcal{S} = f(\mathbb{R}^n), f \quad \textit{regular}\}$$

One-dimensional polynomial images. The following assertions are equivalent:

(i) $\mathcal{S}$ is a polynomial image of $\mathbb{R}^n$ for some $n \geq 1$.

(ii) $\mathcal{S}$ is irreducible, unbounded and $\operatorname{Cl}^{\text{zar}}_{\mathbb{CP}^m}(\mathcal{S})$ is an invariant rational curve such that $\operatorname{Cl}^{\text{zar}}_{\mathbb{CP}^m}(\mathcal{S}) \cap \mathsf{H}_\infty(\mathbb{C}) = \{p\}$ and the germ $\operatorname{Cl}^{\text{zar}}_{\mathbb{CP}^m}(\mathcal{S})_p$ is irreducible.

If that is the case, $\operatorname{p}(\mathcal{S}) \leq 2$. In addition, $\operatorname{p}(\mathcal{S}) = 1 \iff \mathcal{S}$ is closed in $\mathbb{R}^m$.

One-dimensional regular images. The following assertions are equivalent:

(i) $\mathcal{S}$ is a regular image of $\mathbb{R}^n$ for some $n \geq 1$.

(ii) $\mathcal{S}$ is irreducible and $\operatorname{Cl}^{\text{zar}}_{\mathbb{RP}^m}(\mathcal{S})$ is a rational curve.

If that is the case, then $\operatorname{r}(\mathcal{S}) \leq 2$. In addition, $\operatorname{r}(\mathcal{S}) = 1 \iff$ either $\operatorname{Cl}_{\mathbb{RP}^m}(\mathcal{S}) = \mathcal{S}$, or $\operatorname{Cl}_{\mathbb{RP}^m}(\mathcal{S}) \setminus \mathcal{S} = \{p\}$ and the analytic closure of the germ $\mathcal{S}_p$ is irreducible.

Curiosity. *There is no $\mathcal{S}$ with $\operatorname{p}(\mathcal{S}) = 2$ and $\operatorname{r}(\mathcal{S}) = 1$.*

5. Work in progress: Shiota's conjecture for the Nash case

A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is Nash if each component of f is a Nash function, that is, a smooth function with semialgebraic graph. Let $\mathcal{S} \subset \mathbb{R}^m$ be a semialgebraic set of dimension d.

Shiota's conjecture. $\mathcal{S}$ is a Nash image of $\mathbb{R}^d$ if and only if $\mathcal{S}$ is pure dimensional and there exists an analytic path $\alpha : [0,1] \to \mathcal{S}$ whose image meets all connected components of the set of regular points of $\mathcal{S}$.

Consequences. *Two relevant consequences of the positive solution to Shiota's conjecture are: Arc-symmetric sets, Elimination of inequalities.*

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Pawel Gładki: Witt equivalence of function fields over global and local fields.

In this talk we investigate the Witt equivalence of certain types of fields. In particular, we show that for two Witt equivalent function fields over global fields there is a natural bijection between certain Abhyankar valuations of these fields, that corresponds to Witt equivalence of respective residue fields. We also examine to what extent this result carries over to local fields.

Charu Goel: The analogue of Hilbert's 1888 theorem for even symmetric forms.

Sums of squares representations of polynomials is of fundamental importance in real algebraic geometry. In 1888, Hilbert gave a complete characterisation of the pairs $(n, 2d)$ for which a n-ary $2d$-ic form non-negative on $\mathbb{R}^n$ can be written as sums of squares of other forms, namely $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ if and only if $n = 2, d = 1$, or $(n, 2d) = (3, 4)$, where $\mathcal{P}_{n,2d}$ and $\Sigma_{n,2d}$ are respectively the cones of positive semidefinite (psd) and sum of squares (sos) forms (real homogenous polynomials) of degree $2d$ in n variables. This talk presents our analogue of Hilbert's characterisation under the additional assumptions of even symmetry on the given form.

We show that for the pairs $(n, 2d) = (3, 2d)_{d\geq 6}, (n, 8)_{n\geq 5}, (n, 2d)_{n\geq 4, d=5,6}$ and $(n, 2d)_{n\geq 4, d\geq 7}$ there are even symmetric psd not sos n-ary $2d$-ic forms. We establish that an even symmetric n-ary $2d$-ic psd form is sos if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (n, 4)_{n\geq 3}$ or $(n, 2d) = (3, 8)$.

This is joint work with S. Kuhlmann and B. Reznick.

Ilia Itenberg: Hurwitz numbers for real polynomials

We introduce a signed count of real polynomials which gives rise to a real analog of Hurwitz numbers in the case of polynomials. The invariants obtained allow one to show the abundance of real solutions in the corresponding enumerative problems: in many cases the number of real solutions is asymptotically equivalent (in the logarithmic scale) to the number of complex solutions. This is a joint work with Dimitri Zvonkine.

Maria Infusino: Moment problem for symmetric algebras of locally convex spaces

This talk aims to introduce an infinite dimensional version of the classical moment problem, namely the full moment problem on a locally convex topological space, and exploring certain instances of this problem. Given a locally convex topology τ on a real vector space V and a linear functional L on the symmetric algebra $S(V)$, the question addressed is whether

L can be represented as an integral w.r.t. a non-negative Radon measure supported on the algebraic dual V^* of V. I present in particular a recent joint work with M. Ghasemi, S. Kuhlmann and M. Marshall where we explain how a locally convex topology τ on V naturally extends to a locally multiplicatively convex topology $\overline{\tau}$ on $S(V)$. This allows us to apply some recent results on locally multiplicatively convex topological algebras due to M. Ghasemi, S. Kuhlmann and M. Marshall to obtain representations of $\overline{\tau}$-continuous positive semidefinite linear functionals $L : S(V) \to \mathbb{R}$ as integrals w.r.t. uniquely determined Radon measures μ supported on special sorts of closed balls in the topological dual space V' of V.

Zur Izhakian: Supertropical algebra and representations

Tropical mathematics is carried out over idempotent semirings, a weak algebraic structure that on one hand allows descriptions of objects having a discrete nature, but on the other hand, its lack of additive inverse prevents the access to basic mathematical notions. These drawbacks are overcome by the use of a supertropical semiring – a "cover" semiring structure having a special distinguished ideal that plays the role of the zero element in classical mathematics. This semiring structure is rich enough to permit a systematic development of tropical algebraic theory, yielding analogues to many important results and notions from classical commutative algebra. Supertropical algebra provides a suitable algebraic framework that enables natural realizations of matroids and simplicial complexes, as well as representations of semigroups.

Franziska Jahnke: Dp-minimal ordered fields

We consider dp-minimal ordered fields and show that they are either real closed or admit a non-trivial definable valuation in the language of ordered fields. Moreover, we show that every dp-minimal valued field is henselian. This allows us to give a classification of dp-minimal ordered fields as a certain class of power series fields over the reals.
(Joint work with Pierre Simon and Erik Walsberg)

Tobias Kaiser: Lebesgue motivic invariants

We use our recently developed Lebesgue measure and integration theory on arbitrary real closed fields to define new invariants of motivic style for semialgebraic sets over the reals. For this purpose we work with the field of real Puiseux series as arc space.

Jochen Königsmann: Galois codes for arithmetic and geometry through the power of valuation theory.

We give a survey on recent advances in Grothendiek's program of anabelian geometry to characterize arithmetic and geometric objects in Galois theoretic terms. Valuation theory plays a key role in these developments, thus confirming its well deserved place in mainstream mathematics.

Katarzyna Kuhlmann: Orderings and R-places of function fields.

It is well known that the space $M(F)$ of real-valued places of an algebraic function field F in one variable over an archimedean real closed field K is a disjoint union of finitely many simple closed curves. In this case every $\mathbb{R}$-place of F can be regarded as a real point of a complete real algebraic curve. For a non-archimedean real closed field K the structure of $M(F)$ is not well understood yet.

However, we have a description of the space $M(F)$ where F is a rational function field over K. In this case the space $X(F)$ of orderings of F is homeomorphic to the ordered set $C(K)$ of cuts in K and $M(K)$ is its quotient space, which we obtain in the following way. The real closed field K is an ultrametric space with an ultrametric distance u determined by a natural valuation v of K, i.e., $u(a,b) = v(b-a)$. Every ultrametric ball in K determines two cuts of K in a natural way. By identifying cuts determined by the same ultrametric ball we obtain an equivalence relation $\sim$ on $C(K)$. Then $M(K) \cong C(K)/\!\sim$. By this observation we can see a fractal-like structure of $M(F)$ with lots of self-similarities.

In the present talk we will generalize these results to any algebraic function field F in one variable over a non-archimedean real closed field K. Here F can be seen as a field of rational functions on some smooth irreducible complete algebraic curve over K.

In the papers *On algebraic curves over real closed fields* (I and II) M. Knebusch described the structure of the set γ of real points of a smooth irreducible complete algebraic curve over K. We use these results to define cuts on γ. Then we show that there is a homeomorphism between the space of all cuts on γ with its natural topology and the space of orderings of F.

Subsequently we give a necessary and sufficient condition for cuts to induce the same $\mathbb{R}$-place of F. This requires studying the ultrametric properties of the space K^n.

Franz-Viktor Kuhlmann: Symmetrically complete ordered fields.

Which are the ordered fields in which every descending chain of nonempty closed intervals has a nonempty intersection? This is true for the reals, but are there other ordered fields that have this property? Or does it imply that the field is cut complete and hence isomorphic to the reals? The clue is that you can zoom in on a cut using a descending chain of nonempty closed intervals only if the cofinality of the left cut set is equal to the coinitiality of the right cut set; we call such cuts "symmetrical". Hence an ordered field has the above property if and only if it has no symmetrical cuts; we call such fields (and other ordered structures) “symmetrically complete”. In [1] we have shown that such fields satisfy an interesting fixed point theorem; it is quite similar to Banach’s fixed point theorem although the fields need not be archimedean ordered.

Saharon Shelah [3] showed in 2004 that every ordered field can be extended to a symmetrically complete field. In joint work [2] with Katarzyna and Saharon, we have given a direct construction of such fields using power series. We construct first symmetrically complete linear orderings, then symmetrically complete ordered Hahn groups having these as their index sets and archimedean components isomorphic to the reals. In fact, we also show that

every symmetrically complete ordered abelian group must be of this form, hence in particular divisible. Similarly, the power series fields over the reals with symmetrically complete value groups (plus a minor additional condition) are symmetrically complete ordered fields. Conversely, every symmetrically complete ordered field is of this form, hence in particular real closed.
References

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(Joint work with Katarzyna Kuhlmann.)

Salma Kuhlmann: Real Closed Fields and Models of Peano Arithmetic

We say that a real closed field is an IPA-real closed field if it admits an integer part (IP) which is a model of Peano Arithmetic (PA). In [2] we prove that the value group of an IPA-real closed field must satisfy very restrictive conditions (i.e. must be an exponential group in the residue field, in the sense of [4]). Combined with the main result of [1] on recursively saturated real closed fields, we obtain a valuation theoretic characterization of countable IPA-real closed fields. Expanding on [3], we conclude the talk by considering recursively saturated o-minimal expansions of real closed fields and their IPs.

References:
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Henri Lombardi: On the constructive algebra of real numbers

This talk is a survey of natural questions (with few answers) arising when one wants to study algebraic properties of real numbers in a constructive setting. Why studying constructive real algebra?
A first reason is that constructive real algebra is not well understood! Constructive analysis is much more developed.
From a constructive point of view, real algebra is far away from the theory of discrete real closed fields (which was settled by Artin in order to understand real algebra in the framework of classical logic). Most algorithms for discrete real closed fields fail for Dedekind real numbers, because we have no sign test for real numbers.

Another reason is that within constructive analysis, it should be interesting to drop dependent choices. A study of real algebra without dependent choices could help.
Last but not least, understanding constructive real algebra should be a first important step towards a constructive version of O-minimal structures.

Francisco Miraglia: Theory of Quadratic Forms and its Applications in Ring Theory

The talk will consist of two parts. In the first we shall recall some of the distinct approaches to an abstract theory of quadratic forms: Abstract Witt Rings ([Mar1]), Abstract Order Spaces ([Mar2]) and Linked Quaternionic Maps ([MY]). All of these axiomatic treatments are equivalent to Special Groups ([DM1]) in the sense that models of any one system can be canonically and functorially constructed from models of the others.

The second part of the talk we will concentrate on joint work with M. Dickmann, the theory of *Faithfully Quadratic Rings*, to appear in the Memoirs of the AMS. In all that follows the word "ring" stands for a commutative, unitary, semireal (-1 is not a sum of squares) ring, wherein 2 is invertible; if A is a ring, we understand by "quadratic form over A" a diagonal quadratic form with unit coefficients in A.

We give a unified treatment for pairs $\langle A, T \rangle$, where A is a ring and T is either a proper preorder of A ($\langle A, T \rangle$ is a p-ring) or $T = A^2$.

In the ring context, we employ an extension of the classical notion of (matrix) isometry to that of T-isometry, bringing into play the preorder T in the case of preordered rings (if B is a subset of a ring, write $B^\times$ for the units in B), given by:

(I) *Two n-dimensional quadratic forms $\varphi = \sum_{i=1}^n a_i X_i^2$, $\psi = \sum_{i=1}^n b_i X_i^2$, with a_i, $b_i \in A^\times$ are* T-isometric, $\varphi \approx_T \psi$, *if there is a sequence $\varphi_0, \varphi_1, \ldots, \varphi_k$ of n-dimensional diagonal forms over $A^\times$, so that $\varphi = \varphi_0$, $\psi = \varphi_k$ and for every $1 \le i \le k$, either there is a matrix $M \in \mathrm{GL}_n(A)$ such that $\varphi_i = M\varphi_{i-1}M^t$, or there are $t_1, \ldots, t_n \in T^\times$ such that $\varphi_i = \langle t_1x_1, \ldots, t_nx_n \rangle$ and $\varphi_{i-1} = \langle x_1, \ldots, x_n \rangle$.*

Note that if $T = A^2$, then T-isometry is classical matrix isometry.

Next, to a pair $\langle A, T \rangle$ as above (i.e., a p-ring or $T = A^2$) we associate a structure, $G_T(A)$, whose domain is $A^\times/T^\times$, endowed with the product operation induced by $A^\times$, together with a binary relation $\equiv_{G_T(A)}$, defined on ordered pairs of elements of $A^\times/T^\times$, called *binary isometry*, and having $-1 = -1/T^\times$ as a distinguished element. The structure $\langle G_T(A), \equiv_{G_T(A)}, -1 \rangle$ is not quite a special group in the sense of [DM1], but it satisfies some of its axioms, constituting a *proto-special group*. The ring-theoretic approach, based on the definition in (I), and the formal approach (via $G_T(A)$), though related, *are far from identical.*

We shall then introduced three axioms formulated in terms of T-isometry and the *value representation relation* D_T^v on $\langle A, T \rangle$, defined by: for a, $b_1, \ldots, b_n$ in $A^\times$

$$a \in D_T^v(b_1, \ldots, b_n) \quad \Leftrightarrow \quad \exists\, t_1, \ldots, t_n \in T \text{ such that } \quad a = \sum_{i=1}^n t_i b_i.$$

These axioms express elementary properties of value representation, well-known in the classical theory of quadratic forms over fields. We then show that, when satisfied by $\langle A, T \rangle$, these axioms are sufficient $-$ and under mild assumptions, also necessary $-$ to ensure identity between the ring-theoretic and formal approaches, that is:
(II) The structure $G_T(A)$ is a special group.
(III) T-isometry and value representation in $\langle A, T \rangle$ are faithfully coded by the corresponding formal notions in $G_T(A)$.

We call ***T*-faithfully quadratic** any pair $\langle A, T \rangle$ verifying these axioms. Although we shall focus in the case of rings, this setting, as well as the consequences (II) and (III), apply, more generally, to forms with entries in certain subgroups of $A^{\times}$, called *T-subgroups*.

It also worth noticing that, under these axioms, if $\langle A, T \rangle$ is a p-ring, an analog of the classical Pfister local-global principle holds for T-isometry.

Quadratic faithfulness of $\langle A, A^2 \rangle$ ensures that the mod 2 Milnor K-theory of A obtained from that in [Gu] coincides with the K-theory of the special group $G(A)$ as defined in [DM2]. In fact, only the simplest of our axioms is needed here.

The axioms for T-quadratic faithfulness can be formalized by first-order sentences in the language of unitary rings (consisting of $+$, $\cdot$, 0, 1, 1) augmented by a unary predicate symbol for T (not needed if $T = A^2$). It is shown that there is a Horn-geometric axiomatization of the the notion of T-quadratic faithfulness, T a preorder, which can be explicitly stated in case $T = A^2$, guaranteeing the preservation T-quadratic faithfulness under (right-directed) inductive limits and arbitrary products.

We shall state results showing that certain outstanding classes of p-rings $\langle A, T \rangle$ are T-faithfully quadratic: rings with many units (or local-global rings), reduced f-rings and Archimedean p-rings with bounded inversion.

The considerable effort demanded by these proofs is rewarded by the significance of the results thus reaped, establishing that the theory of diagonal quadratic forms with invertible entries over several classes of p-rings, important in mathematical practice and although far from being fields, possess many of the pleasant properties of quadratic form theory previously known to hold only in the case of formally real fields.

References

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Tim Netzer: Proving Kazhdan's Property (T) with sums of squares techniques

Methods from Real Algebra and Geometry have useful applications in various branches of mathematics. I will report on another such application, coming from group theory. Kazhdan's Property (T) is a deep and often hard to prove property of a group, involving all unitary representations on Hilbert spaces. Surprisingly, it can also be stated in terms of sums of squares in the group algebra, and is thus accessible to optimization methods. With this approach we give a new and easy proof of property (T) for $\mathrm{SL}_3(\mathbb{Z})$, and at the same time improve the so-called spectral gap significantly.

Daniel Perrucci: Elementary recursive degree bounds for the Positivstellensatz, Hilbert's 17th problem and the Real Nullstellensatz.

Hilbert's 17th problem is to express a non-negative polynomial as a sum of squares of rational functions. Artin's original proof is non-constructive and gives no information on degree bounds. A more general problem is to give an identity which certifies the unrealizability of a system of polynomial equations and inequalities. The existence of such an identity is guaranteed by the Positivstellensatz, and Hilbert's 17th problem as well as the Real Nullstellensatz follow easily from such identity. In this talk, we explain a new constructive proof which provides elementary recursive bounds for the Positivstellensatz, Hilbert's 17th problem, and the Real Nullstellensatz, namely a tower of five levels of exponentials.

This is joint work with Henri Lombardi (Université de Franche-Comté, France) and Marie-Françoise Roy (Université de Rennes 1, France)

Albrecht Pfister: An elementary proof of Hilbert's theorem on ternary quartics

I report on a joint paper with Claus Scheiderer which appeared in the Journal of Algebra (2012). In the first part of my talk there will be some comments on Hilbert's paper of 1888 and on the later development. Hilbert's "work" is short and ingenious but also far ahead of its time because it uses advanced methods from topology and algebraic geometry. Some topics like "dimension theory" have been completed about "50 years" later, all the "detailed proofs" of his theorem even appeared "after 1988". In the second part I will outline the main ingredients of our proof.

Daniel Plaumann: Positive polynomials according to Murray Marshall

In this survey talk, we will give an overview of the theory of positive polynomials, focussing on some of Murray Marshall's numerous contributions.

Bruce Reznick: Ternary forms with lots of zeros

In 1980, Choi, Lam and the speaker proved that if a psd ternary sextic has more than 10 zeros, then it has a square indefinite factor, infinitely many zeros and is sos. We discuss the situation in higher degree, including examples of Harris.
This is, in part, joint work with Greg Blekherman.

Cordian Riener: Topological complexity of symmetric semi-algebraic sets

This talk will present results on the structure of cohomology modules of symmetric (as well as multi-symmetric) real varieties and semi-algebraic sets. More precisely, we will show bounds on the number of irreducible representations of the symmetric group occurring in the isotypic decomposition of the cohomology modules of the sets in question, which are polynomial in the number of variables, as well as polynomial bounds on the corresponding multiplicities. As an application we will give some improvements on bounds for the Betti numbers of images under projections of (not necessarily symmetric) bounded real algebraic sets. (Based on joint work with Saugata Basu.)

Jean-Philippe Rolin: Logarithmico-exponential series and fractal analysis.

We will describe a new approach to the study of discrete dynamical systems on the real line, which consists in considering their orbits as "fractal objetcs". In particular, the formal classification of analytic systems can be reproven with this method. We will also explain the main lines of a program devoted to the study of some non analytic systems. These are generated by maps which admit a specific type of transseries (Dulac's transseries) as asymptotic expansions.

Claus Scheiderer: Extremal nonnegative forms.

A nonnegative form $f(x_1, \ldots, x_n)$ with real coefficients is said to be extremal if it can be written as a sum of two nonnegative forms only in a trivial way. Every nonnegative form is a finite sum of extremal forms. We will survey known results about extremal forms and will then discuss methods for constructing ternary such forms. For degree six, our approach yields essentially all extremal forms.

Konrad Schmüdgen: Truncated moment problem and positive polynomials.

Let $\mathtt{s}N$ be a finite subset of $\mathbb{N}_0^d$ such that $0 \in \mathtt{s}N$, $\mathtt{s}A$ be the span of monomials $x^\alpha, \alpha \in \mathtt{s}N$, and K be a closed subset of $\mathbb{R}^d$. The truncated K-moment problem (TMP) asks:

Given a linear functional L on $\mathtt{s}A$, when does there exist a positive Borel measure μ on $\mathbb{R}^d$ supported by K such that

$$L(f) = \int f(x) d\mu(x) \quad \text{for} \quad f \in \mathtt{s}A?$$

The talk is mainly a survey of known results on the (TMP), but it presents also some new results from recent work by the speaker and Philipp di Dio.

If the (TMP) has a solution, it has always a k-atomic solution, where $k \leq \dim \mathtt{s}A$. A number of existence criteria are reviewed (flat extension theorem of R. Curto and L. Fialkow, characterizations in terms of positive polynomials, projective versions). Several special questions are discussed (determinacy, maximal mass representations, minimal atomic representations, evaluation polynomials).

Hans Schoutens: O-minimalism: the first-order properties of o-minimality.

O-minimalism is the first-order theory of o-minimal structures, an important class of models of which are the ultraproducts of o-minimal structures. A complete axiomatization of o-minimalism is not known, but many results are already provable in the weaker theory DCTC given by definable completeness and type completeness (a small extension of local o-minimality). In DCTC, we can already prove how many results from o-minimality (dimension theory, monotonicity, Hardy structures) carry over to this larger setting upon replacing 'finite' by 'discrete, closed and bounded'. However, even then cell decomposition might fail, giving rise to a related notion of tame structures. Some new invariants also come into play: the Grothendieck ring is no longer trivial and the definable, discrete subsets form a totally ordered structure induced by an ultraproduct version of the Euler characteristic. To develop this theory, we also need another first-order property, the Discrete Pigeonhole Principle, which I cannot yet prove from DCTC. Using this, we can formulate a criterion for when an ultraproduct of o-minimal structures is again o-minimal.

Niels Schwartz: The sums of squares in a real field

The sums of squares in a real field K form a semifield. It will be shown how the semifield can be used to analyze the real valuations of K.

Markus Schweighofer: Inclusion of spectrahedra, free spectrahedra and coin tossing.

Spectrahedra are generalizations of (convex) polyhedra sharing many of the good algorithmic features with polyhedra but allowing for roundness in their shapes. Given two spectrahedra in form of a linear matrix inequality, it is in general hard to decide whether one contains the other. To the linear matrix inequalities one can however associate not only scalar solutions but also matrix solutions. This gives rise to so called free spectrahedra. Inclusion of free spectrahedra is in many cases easy to decide, e.g., by using a generalization of the Gram matrix method and a linear Positivstellensatz for symmetric linear matrix polynomials due to Helton, Klep and McCullough . The natural question arises how the inclusion of two free spectrahedra relates to the inclusion of the corresponding classical spectrahedra. Surprisingly, this question is related to very subtle non-trivial properties of Binomial and Beta distributions some of which are known and some of which are new.
(joint work with Bill Helton, Igor Klep and Scott McCullough).

Tamara Servi: Lebesgue integration of oscillating and subanalytic functions

We consider the algebra generated by all global subanalytic functions, their logarithms and the functions exp (if), where f is subanalytic. We aim to understand the nature of the parametric integrals of the functions in the algebra. Is this family stable under integration?. What is the nature of the locus of integrability?. We show that the answer to these questions can be obtained by adding certain "transcendental" functions to the original algebra.
Joint work with R. Cluckers, G. Comte, D. Miller and J.-P. Rolin.

Rainer Sinn: Gram Spectrahedra.

Gram spectrahedra parametrize the set of all sum of squares representations of a given polynomial. They are convex semi-algebraic sets and we will consider them from the point of view of convex algebraic geometry. We present results on their algebraic boundary and we discuss ranks of extreme points and rationality questions.

Mark Spivakovsky: The problem of local uniformization in arbitrary characteritic.

Let k be a field and R a noetherian k-algebra without zero divisors. Let R_ν be a valuation ring containing R and having the same field of fractions as R. The problem of **local uniformization** of R with respect to ν is one of finding a *smooth* finite type R-algebra R' such that $R \subset R' \subset R_\nu$. The Local uniformization theorem is known when *char* $k = 0$ and is one of the central open problems in the field when *char* $k = p > 0$.

In the first half of the talk we will introduce the formalism of key polynomials for attacking the local uniformization problem.

Let $K \subset K(x)$ be a simple extension of valued fields.

Let ν denote the given valuation of $K(x)$ and $\Gamma = \nu(K(x)^*)$ its value group. Let Γ_+ be the set of non-negative elements of Γ. For a non-negative integer b the symbol ∂_b will stand for the Hasse derivative of order b:

$$\partial_b = \frac{1}{b!}\frac{\partial^b}{\partial x^b}.$$

For a polynomial $f \in K[x]$ put

$$\epsilon(f) = \min_{b\in\mathbb{N}_{>0}} \frac{\nu(f) - \nu(\partial_b f)}{b}.$$

Take an element $\epsilon \in \Gamma_+$. We say that f is a **pseudo-key polynomial of level** ϵ if the following conditions hold:

(1) f is monic

(2) $\epsilon(f) = \epsilon$

(3) for any polynomial $g \in K[x]$ if $\epsilon(g) \geq \epsilon$ then deg $g \geq$ deg f. We will define a subclass of pseudo-key polynomials called **key polynomials** and describe our proposed applications of this formalism to the local uniformization problem.

Bernard Teissier: Valued noetherian local rings versus valued fields

Many important results in the theory of valued fields are inspired by the embedding of the local ring of a plane branch in a valuation ring which is given by the Newton-Puiseux theorem. The valuation of an algebraic function field K is determined by the values it takes on a noetherian ring R contained in the valuation ring and having K as field of fractions. Another approach to study valuations is to study equations defining (a suitable completion of) R and compatible with the valuation in the sense that their initial forms with respect to a natural weight generate the ideal defining the associated graded ring of R with respect to the valuation. The noetherianity of R then provides tools to understand the valuation through the combinatorics of the semigroup of the values it takes on the ring.

Margaret Thomas: Effective Pila–Wilkie bounds for restricted Pfaffian surfaces

The counting theorem of Pila and Wilkie opened up one of the most important developments in applied model theory in recent years. It provides a bound on the density of rational points of bounded height lying on the 'transcendental parts' of sets definable in o-minimal expansions of the real field, a result which has had several stunning number theoretic applications (e.g. to the Manin-Mumford and André-Oort Conjectures). However, the proof of the theorem is not effective: it does not give a procedure which, given a definable set, will compute the Pila–Wilkie bound for that set. This of course constrains the effectivity of its applications. I will discuss some recent progress made towards finding an effective version of the Pila–Wilkie Theorem in certain cases

Marcus Tressl: Model theory of real closed rings.

A real closed ring (in the sense of Niels Schwartz) is a commutative unital ring on which continuous semi-algebraic functions operate in a natural way. For example rings of continuous real-valued functions on a topological space and convex subrings of real closed fields are real closed (a precise definition will be given in the talk).

A field is real closed as a ring just if it is a real closed field. However, unlike in the field case, real closed rings generally have a complicated first order theory. For example the ring of continuous functions on the real line interprets the projective hierarchy.

On the other hand, real closed rings are algebraically well behaved; for example: Basic operations from commutative algebra can be done within the category of real closed rings; every ring has a unique "real closure"; all real closed rings that are a domain, are local henselian rings.

I will give a brief introduction to the model theory of real closed rings and then focus on a few special cases.

Lou van den Dries: The ordered differential field of transseries

The field of Laurent series (with real coefficients, say) has a natural derivation but is too small to be closed under integration and other natural operations such as taking logarithms of positive elements. The field has a natural extension to a field of generalized series, the ordered differential field of transseries, where these defects are remedied in a radical way. I will sketch this field of transseries. Recently it was established (Aschenbrenner, Van der Hoeven, vdD) that the differential field of transseries also has very good model theoretic properties. I hope to discuss this in the later part of my talk.

Grey Violet: Geometry of univariate stability: continuity argument, symmetric products and stability theories.

noindent The concept of stable polynomial or, speaking in a more general way, the concept of polynomial root clustering is one the main tools used in different parts of control theory. The geometric approach to parametric stability problems called D-decomposition or root invariance decomposition is one of the fundamental methods in that field. Despite this, the geometry of parametric stability problems is still poorly understood even for such most basic examples as a PID-controller.

Our goal is to provide a general real algebraic geometric framework for different definitions of polynomial stability based on careful analysis of root-coefficient correspondence via symmetric products and stratifications on root and coefficients spaces produced by different definitions of stability.

We examine different aspects of the geometric structure of those stratifications and provide a natural explanation for the distinguished position of Schur's stability, Hurwitz's stability and hyperbolicity among all other stability theories.