Whitney Problems Workshop
October 19 - 23, 2015

Alex Brudnyi:

1 - **On Bernstein Classes of Well Approximable Maps**

We study the structure of families of "well approximable" elements of tensor products of Banach spaces including analogs of the classical quasianalytic classes in the sense of Bernstein and Beurling. As in the case of quasianalytic functions, we prove for members of these families variants of the Mazurkiewicz and Markushevich theorems and in some particular cases, if such elements are Banach-valued continuous maps on a compact metric space, estimate massivity of their graphs and level sets.

2 - **On the Sundberg Approximation Theorem**

Let H^∞ be the Banach algebra of bounded holomorphic functions in the open unit disk $\mathbb{D} \subset \mathbb{C}$. We give an alternative proof of Sundberg's theorem on uniform approximation of functions in $BMOA$ by H^∞ functions. Then we generalize this theorem to certain sets of functions in the Nevanlinna class N. In our approach we use a new characterization of meromorphic functions on $\mathbb{D}$ that extend to continuous maps of the maximal ideal space of H^∞ to the Riemann sphere.

Aris Daniilidis: The convex paradigm in optimization: dynamical considerations.

In this talk, we study the asymptotic behavior of continuous and discrete orbits of dynamical systems that are naturally linked to optimization: gradient orbits, steepest descent curves and proximal algorithm. The convex paradigm and the notion of self-contractedness will be the cornerstone of this study.
Based on works in collaboration with O. Ley, S. Sabourau, E. Durand, A. Lemenant and G. David

Thibaut Deheuvels: A transmission problem across a fractal interface

We study some questions of analysis in view of the modeling of tree-like structures. We consider a bidimensional ramified domain whose boundary contains a fractal self-similar set Γ. The domain can be seen as a 2D idealization of the human bronchial tree. We focus on the case when Γ has self-intersections.

We start by giving trace results on the fractal boundary, and Sobolev extension results. The existence of Sobolev extension operators depends on the Hausdorff dimension of the self-intersection of the boundary. We propose a construction based on a Haar wavelet decomposition on the fractal set Γ.

We consider elliptic transmission problems accross the fractal part of the boundary. We introduce approximations of the ramified domain with Lipschitz boundary. We are interested in the Γ-convergence of the energy forms of the approximated problems to the energy form of the problem with fractal interface. We focus on a particular geometry with self-intersections, for which the proof of the Γ-convergence relies on an extension result in the ramified domain. The construction of Sobolev extension operators relies on an extension result by Auscher-Badr in multiple cones.

This is a joint work with Y. Achdou (Université Paris 7).

Charles Fefferman:

1 - Whitney problems survey

This talk surveys some results but emphasizes unsolved problems. Topics include finiteness theorems and algorithms for $\mathcal{C}^m$, analogues for Sobolev spaces, sharp constants, $\mathcal{C}^\infty$ extension, connections to real algebraic geometry, extensions with constraints, fitting manifolds to data,...

2 - Whitney problems and real algebraic geometry

This talk sketches connections between Whitney problems and e.g. the problem of deciding whether a given rational function on $\mathbb{R}^n$ belongs to $\mathcal{C}^m$.

Ariel Herbert-Voss, Matthew Hirn and Frederick McCollum: Computing minimal interpolants in $C^{1,1}(\mathbb{R}^d)$

We consider the following interpolation problem. Suppose one is given a finite set $E \subset \mathbb{R}^d$, a function $f : E \to \mathbb{R}$, and possibly the gradients of f at the points of E as well. We want to interpolate the given information with a function $F \in C^{1,1}(\mathbb{R}^d)$ with the minimum possible value of $\mathrm{Lip}(\nabla F)$. We present practical, efficient algorithms for constructing an F such that $\mathrm{Lip}(\nabla F)$ is minimal, or for less computational effort, within a small dimensionless constant of being minimal.

Ritva Hurri-Syrjänen: On the (q,p)-Poincaré inequality, when $q < p$

We study the question of domains where the classical (p,p)-Poincaré inequality fails for some p, but the (q,p)-Poincaré inequality could hold with some $q < p$.

This talk is based on my joint work with Petteri Harjulehto, Niko Marola, and Antti Vähäkangas, J. Math. Anal. Appl. (2008), Illinois Math. J. (2012) and Manuscripta Math. (2015).

Lizaveta Ihnatsyeva : Measure density and extension of Besov and Triebel–Lizorkin functions

Recently there have been introduced several analogues for Besov spaces and Triebel–Lizorkin spaces in a quite general setting, which, in particular, includes certain topological manifolds, fractals, graphs and Carnot-Carathéodory spaces. Employing one of the available definitions, we study extension domains for Besov-type and for Triebel-Lizorkin type functions in the setting of a metric measure space with a doubling measure; as a special case we obtain a characterization of extension domains for classical Besov spaces defined via L^p-modulus of smoothness. The talk is based on joint work with Toni Heikkinen and Heli Tuominen.

Arie Israel: Interpolation of data in Sobolev spaces

We explain some of the main ingredients in our work on algorithms for Sobolev extension. The talk is based on joint work with Charles Fefferman and Kevin Luli.

Nikos Katzourakis: Vectorial Calculus of Variations in L^∞ and generalised solutions for fully nonlinear PDE systems.

Calculus of Variations in L^∞ has a long history, the scalar case of which was initiated by G.Aronsson in the 1960s and is under active research ever since. Aronsson's motivation to study this problem was related to the optimisation of Lipschitz Extensions of functions. Mathematically, minimising the supremum is very challenging because the equations are non-divergence and highly degenerate. However, it provides more realistic models, as opposed to the classical case of minimisation of the average (integral). However, due to fundamental difficulties, until the early 2010s the field was restricted to the scalar case. In this talk I will discuss the vectorial case, which has recently been initiated by the speaker. The analysis of the L^∞-equations is based on a recently proposed general duality-free PDE theory of generalised solutions for fully nonlinear systems.

Krzysztof Kurdyka: Curve-rational functions

Let X be an algebraic subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$ a semialgebraic function. We prove that if f is continuous rational on each curve $C \subset X$ then: 1) f is arc-analytic, 2) f is continuous rational on X. As a consequence we obtain a characterization of hereditarily rational functions recently studied by J. Kollár and J.K. Nowak,. The last paper is related to a recent work of C. Fefferman and J. Kollar on solutions of linear systems with polynomial coefficients.
Joint work with W. Kucharz

Erwan Y. Le Gruyer: Extremal Extension for m-jets of one variable with range in a Hilbert space

We generalize to Hilbert spaces a theorem of Glaeser concerning minimal Lipschitz extensions of m-jets of one variable with range in $\mathbb{R}$. The results contained in this paper can been seen as a small contribution to the general problem of the minimal Lipschitz extensions from m-jets for a Hilbert space to another Hilbert space.

Fernando Lopez Garcia: A decomposition of functions and weighted Korn inequality on John domains

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $n \geq 2$. For a vector field $\mathbf{u}$ in the Sobolev space $H^1(\Omega)^n$ we denote by $\varepsilon(\mathbf{u})$ the symmetric part of the differential matrix $D\mathbf{u}$ of $\mathbf{u}$. The classical Korn inequality, which has been widely studied for its applications to the linear elasticity equations, states

$$\|D\mathbf{u}\|_{L^2(\Omega)} \leq C\|\varepsilon(\mathbf{u})\|_{L^2(\Omega)},$$

where the vector field $\mathbf{u}$ satisfies $\int_\Omega \frac{\partial \mathbf{u}_i}{\partial x_j} - \frac{\partial \mathbf{u}_j}{\partial x_i} = 0$.

In this talk, we will show a weighted version of Korn inequality on John domains, where the weights are arbitrary nonnegative powers of the distance to the boundary. Notice that the weights may not be in the Muckenhoupt class. We also make an estimation of the constant in the inequality which depends on the power appearing in the weight and a certain geometric condition verified by John domains. In order to prove these results we use a local-to-global technique which is based on a certain decomposition of functions.

Kevin Luli: Interpolation by Nonnegative Functions

Let E be a finite subset of $\mathbb{R}^n$. Given $f : E \to [0,\infty)$, how can we tell whether there exists a nonnegative function $F : \mathbb{R}^n \to [0,\infty)$ such that

$F \in C^m(\mathbb{R}^n)$, $F|_E = f$? In this talk, we will explain a finiteness principle that answers this question. In a nutshell, it states that there exists a constant $k^\#$ depending only on m and n such that

if for any subset $S \subset E$ with at most $k^\#$ points, we can find $F^S \in C^m(\mathbb{R}^n)$ such that $F \geq 0$ and $F^S|_S = f$, then we can find $F : \mathbb{R}^n \to [0,\infty)$ such that

$F \in C^m(\mathbb{R}^n)$ such that $F|_E = f$.

This is a joint work with C. Fefferman and A. Israel.

Andreea Nicoara: Direct proof of termination of the Kohn algorithm in the real-analytic case

In 1979 J.J. Kohn gave an indirect argument that his algorithm terminates on a pseudoconvex real-analytic domain of finite D'Angelo type. I will give a direct argument for the same assertion by constructing subelliptic multipliers that give a subelliptic estimate at each boundary point in terms of Catlin's boundary system at that point. I will also show what else is needed (two ingredients) in order to turn this argument into an effective one that yields a lower bound for the subelliptic gain in terms of the dimension, D'Angelo type, and order of the forms for any pseudoconvex real-analytic domain of finite D'Angelo type. One of the two ingredients is a conjectured effective Nullstellensatz in the real-analytic case that raises interesting real algebraic geometric issues.

Wieslaw Pawlucki: $\mathbb{C}^k$-extendability criterion for functions on a closed set definable in any polynomially bounded o-minimal structure

We generalize to any polynomially bounded o-minimal structure the results of our earlier paper [E. Bierstone, P.D. Milman, W.Pawlucki; Differentiable functions defined in closed sets. A problem of Whitney; Invent. Math. 151 (2003), no. 2, 329–352].

Rafał Pierzchała: Markov-type inequalities

One of the most important polynomial inequalities is the following Markov's inequality.

Theorem (Markov, 1889) *If P is a polynomial of one variable, then*

$$\|P'\|_{[-1,\,1]} \le (\deg P)^2 \|P\|_{[-1,\,1]}\,.$$

Moreover, this inequality is optimal, because for the Chebyshev polynomials T_n ($n \in \mathbb{N}_0$), we have $T_n'(1) = n^2$ and $\|T_n\|_{[-1,\,1]} = 1$.

Recall that

$$T_n(u) = \frac{1}{2}\left[\left(u + \sqrt{u^2-1}\right)^n + \left(u - \sqrt{u^2-1}\right)^n\right].$$

Markov's inequality and its generalizations found many applications in approximation theory, constructive function theory and analysis (for instance, to Whitney-type extension problems), but also in other branches of science (for example, in physics or chemistry). From the point of view of applications, it is important that the constant $(\deg P)^2$ in Markov's inequality grows not too fast (that is, polynomially) with respect to the degree of the polynomial P.

It is natural to ask about similar inequalities if we replace the interval $[-1,\,1]$ by another compact set in $\mathbb{R}^N$ or $\mathbb{C}^N$. In the talk, we will address this issue. In particular, we will give a solution to an old problem, studied among others by Baran and Pleúniak, and concerning the invariance of Markov's inequality under polynomial mappings.

Pavel Shvartsman: A Whitney-type extension theorem for jets generated by Sobolev functions

Let E be a closed subset of R^n. For each $p > n$ and each positive integer m we give an intrinsic characterization of the restrictions $\{D^\alpha F|_E : |\alpha| \le m\}$ to E of m-jets generated by functions F from the Sobolev space $W_p^{m+1}(R^n)$.

Our trace criterion is expressed in terms of variations of corresponding Taylor's remainders of m-jets evaluated at a certain family of "well separated" two point subsets of E. For $p = \infty$ this criterion coincides with the classical Whitney-Glaeser extension theorem for m-jets.

Ignacio Uriarte-Tuero: Two weight norm inequalities for singular and fractional integral operators in $\mathbb{R}^n$

I will report on recent advances on the topic, related to proofs of T1 type theorems in the two weight setting for Calderón-Zygmund singular and fractional integral operators, with side conditions, and related counterexamples. In particular, I will report on the $T1$ two weight theorem for the Riesz and Cauchy transforms when one measure is supported on a $C^{1,\delta}$ curve. Joint work with Eric Sawyer and Chun-Yen Shen.

Dmitri Yafaev: Rational approximation of singular functions

We consider functions ω on the unit circle $\mathbf{T}$ with a finite number of logarithmic singularities. We study the approximation of ω by rational functions. We find an asymptotic formula for the distance in the BMO norm between ω and the set of rational functions of degree n as $n \to \infty$. Our approach relies on the Adamyan-Arov-Krein theorem and on the study of the asymptotic behaviour of singular values of Hankel operators.

Nahum Zobin: Some duality considerations in Whitney problems

We discuss some dualities relevant to Whitney problems and their applications.