

Commutator criteria for strong mixing

Rafael Tiedra de Aldecoa

Pontifical Catholic University of Chile

CIRM, September 2015

Table of Contents

- 1 Strong mixing
- 2 Commutators
- 3 Discrete groups
- 4 Continuous groups
- 5 Time changes of horocycle flows
- 6 References

Strong mixing

A unitary operator U in a Hilbert space \mathcal{H} is a surjective isometry:

$$U^*U = UU^* = 1.$$

Example (Discrete group of unitary operators)

If U is a unitary operator in a Hilbert space \mathcal{H} ,

$$U_n := U^n, \quad n \in \mathbb{Z},$$

defines a discrete 1-parameter group of unitary operators.

Example (Continuous group of unitary operators)

If H is a self-adjoint operator in a Hilbert space \mathcal{H} , then

$$U_t := e^{-itH}, \quad t \in \mathbb{R},$$

defines a strongly continuous 1-parameter group of unitary operators.

Example (Koopman operator)

If $T : X \rightarrow X$ is an automorphism of a probability space (X, μ) , then the Koopman operator

$$U_T : L^2(X, \mu) \rightarrow L^2(X, \mu), \quad \varphi \mapsto \varphi \circ T,$$

is a unitary operator.

Ergodicity, weak mixing and strong mixing of an automorphism $T : X \rightarrow X$ are expressible in terms of the Koopman operator U_T :

- T is ergodic iff 1 is a simple eigenvalue of U_T .
- T is weakly mixing iff U_T has purely continuous spectrum in $\{\mathbb{C} \cdot \mathbf{1}\}^\perp$.
- T is strongly mixing iff

$$\lim_{N \rightarrow \infty} \langle \varphi, (U_T)^N \varphi \rangle = 0 \quad \text{for all } \varphi \in \{\mathbb{C} \cdot \mathbf{1}\}^\perp.$$

$\text{a.c. spectrum in } \{\mathbb{C} \cdot \mathbf{1}\}^\perp \Rightarrow \text{strong mixing} \Rightarrow \text{weak mixing} \Rightarrow \text{ergodicity}$
--

Commutators

- \mathcal{H} , Hilbert space with norm $\|\cdot\|$ and scalar product $\langle \cdot, \cdot \rangle$
- $\mathcal{B}(\mathcal{H})$, bounded linear operators on \mathcal{H}
- A , self-adjoint operator in \mathcal{H} with domain $\mathcal{D}(A)$

Definition

An operator $S \in \mathcal{B}(\mathcal{H})$ satisfies $S \in C^k(A)$ if

$$\mathbb{R} \ni t \mapsto e^{-itA} S e^{itA} \in \mathcal{B}(\mathcal{H})$$

is strongly of class C^k .

$S \in C^1(A)$ if and only if

$$|\langle A\varphi, S\varphi \rangle - \langle \varphi, SA\varphi \rangle| \leq \text{Const.} \|\varphi\|^2 \quad \text{for all } \varphi \in \mathcal{D}(A).$$

The operator corresponding to the continuous extension of the quadratic form is denoted by $[S, A]$, and one has

$$[iS, A] = s- \frac{d}{dt} \Big|_{t=0} e^{-itA} S e^{itA} \in \mathcal{B}(\mathcal{H}).$$

Definition

A self-adjoint operator H in \mathcal{H} is of class $C^k(A)$ if $(H - z)^{-1} \in C^k(A)$ for some $z \in \mathbb{C} \setminus \sigma(H)$.

If H is of class $C^1(A)$, then

$$[A, (H - z)^{-1}] = (H - z)^{-1} [H, A] (H - z)^{-1},$$

with $[H, A] \in \mathcal{B}(\mathcal{D}(H), \mathcal{D}(H)^*)$ the operator corresponding to the continuous extension to $\mathcal{D}(H)$ of the quadratic form

$$\mathcal{D}(H) \cap \mathcal{D}(A) \ni \varphi \mapsto \langle H\varphi, A\varphi \rangle - \langle A\varphi, H\varphi \rangle \in \mathbb{C}.$$

Discrete groups

Theorem (Strong mixing for discrete groups)

Let U and A be a unitary and a self-adjoint operator in \mathcal{H} , with $U \in C^1(A)$. Assume that

$$D := \text{s-lim}_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n ([A, U] U^{-1}) U^{-n}$$

exists, and suppose that $\eta(D)\mathcal{D}(A) \subset \mathcal{D}(A)$ for each $\eta \in C_c^\infty(\mathbb{R} \setminus \{0\})$.

Then,

- (a) $\lim_{N \rightarrow \infty} \langle \varphi, U^N \psi \rangle = 0$ for each $\varphi \in \ker(D)^\perp$ and $\psi \in \mathcal{H}$,
- (b) $U|_{\ker(D)^\perp}$ has purely continuous spectrum.

- D is bounded and self-adjoint because it is the strong limit of bounded self-adjoint operators.
- $\eta(D)$ with $\eta \in C_c^\infty(\mathbb{R} \setminus \{0\})$ is well-defined by functional calculus.
- $DU^n = U^n D$ for each $n \in \mathbb{Z}$. So, $\ker(D)^\perp$ is a reducing subspace for U , and $U|_{\ker(D)^\perp}$ is a unitary operator.
- Point (b) is a simple consequence of point (a).
- Point (b) could be compared with the Virial theorem for unitary operators (“ $C^1(A) + \dots$ ” \Rightarrow continuous spectrum).

Sketch of the proof of (a). Since $U \in C^1(A)$, one has $U^N \in C^1(A)$ with

$$\begin{aligned}
 [A, U^N] &= \sum_{n=0}^{N-1} U^{N-1-n} [A, U] U^n \\
 &= \left(\sum_{n=0}^{N-1} U^{N-1-n} ([A, U] U^{-1}) U^{n+1-N} \right) U^N \\
 &= \left(\sum_{n=0}^{N-1} U^n ([A, U] U^{-1}) U^{-n} \right) U^N \\
 &= ND_N U^N
 \end{aligned}$$

and

$$D_N := \frac{1}{N} \sum_{n=0}^{N-1} U^n ([A, U] U^{-1}) U^{-n}.$$

So, we have for $\varphi = \eta(D)\varphi$ with $\varphi \in \mathcal{D}(A)$ and $\psi \in \mathcal{D}(A)$

$$\begin{aligned}
 & |\langle \varphi, U^N \psi \rangle| \\
 &= |\langle DD^{-1}\eta(D)\varphi, U^N \psi \rangle| \\
 &\leq |\langle (D_N - D)D^{-1}\eta(D)\varphi, U^N \psi \rangle| + |\langle D^{-1}\eta(D)\varphi, D_N U^N \psi \rangle| \\
 &\leq \|(D_N - D)D^{-1}\eta(D)\varphi\| \cdot \|\psi\| + \frac{1}{N} |\langle D^{-1}\eta(D)\varphi, [A, U^N]\psi \rangle| \\
 &= \|(D_N - D)D^{-1}\eta(D)\varphi\| \cdot \|\psi\| \\
 &\quad + \frac{1}{N} |\langle AD^{-1}\eta(D)\varphi, U^N \psi \rangle - \langle D^{-1}\eta(D)\varphi, U^N A \psi \rangle| \\
 &\leq \|(D_N - D)D^{-1}\eta(D)\varphi\| \cdot \|\psi\| \\
 &\quad + \frac{1}{N} \left(\|AD^{-1}\eta(D)\varphi\| \cdot \|\psi\| + \|D^{-1}\eta(D)\varphi\| \cdot \|A\psi\| \right),
 \end{aligned}$$

which goes to 0 as $N \rightarrow \infty$.

Remark

If $\sum_{N \geq 1} \|(D_N - D)\varphi\|^2 < \infty$ for suitable $\varphi \in \mathcal{H}$, then

$$\sum_{N \geq 1} |\langle \varphi, U^N \varphi \rangle|^2 < \infty \quad \text{for all } \varphi \in \ker(D)^\perp,$$

and $U|_{\ker(D)^\perp}$ has purely a.c. spectrum.

Example (Skew products of compact Lie groups)

...

Continuous groups

Theorem (Strong mixing for continuous groups)

Let H and A be self-adjoint operators in \mathcal{H} , with $(H - i)^{-1} \in C^1(A)$. Assume that

$$D := \text{s-lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t ds e^{isH} (H + i)^{-1} [iH, A] (H - i)^{-1} e^{-isH}$$

exists, and suppose that $\eta(D)\mathcal{D}(A) \subset \mathcal{D}(A)$ for each $\eta \in C_c^\infty(\mathbb{R} \setminus \{0\})$. Then,

- (a) $\lim_{t \rightarrow \infty} \langle \varphi, e^{-itH} \psi \rangle = 0$ for each $\varphi \in \mathcal{H}$ and $\psi \in \ker(D)^\perp$,
- (b) $H|_{\ker(D)^\perp}$ has purely continuous spectrum.

Example (Canonical commutation relation)

Assume that $(H - i)^{-1} \in C^1(A)$ with $[iH, A] = 1$. Then, for all $t > 0$

$$D_t = \frac{1}{t} \int_0^t ds e^{isH} (H + i)^{-1} [iH, A] (H - i)^{-1} e^{-isH} = (H^2 + 1)^{-1} = D$$

and $\ker(D) = \{0\}$. So, the theorem implies that H has purely a.c. spectrum. In fact, we have in this case the Weyl commutation relation

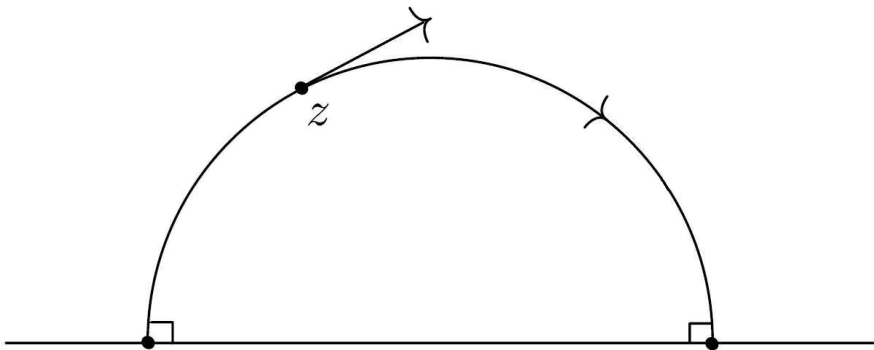
$$e^{-itA} e^{isH} e^{itA} = e^{ist} e^{isH}, \quad s, t \in \mathbb{R}.$$

Thus, Stone-von Neumann theorem implies that H has Lebesgue spectrum with uniform multiplicity.

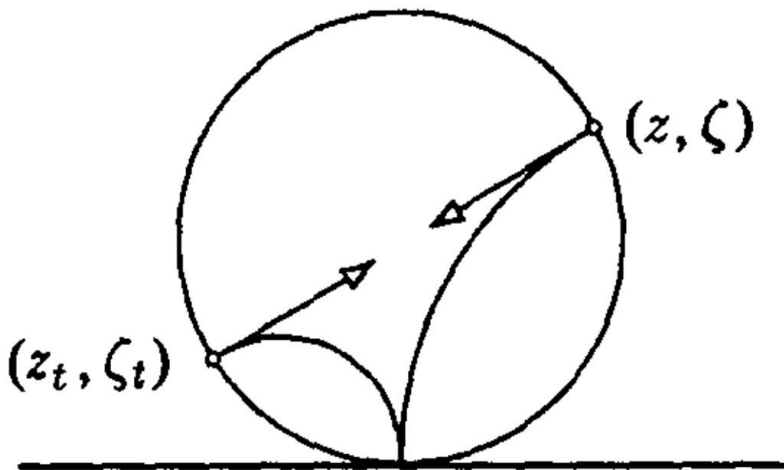
Example (Time changes of horocycle flows).

- Σ , compact Riemannian surface of constant negative curvature
- $M := T^1\Sigma$, unit tangent bundle of Σ
 (M is a compact 3-manifold with probability measure μ ,
 $M \simeq \Gamma \backslash \mathrm{PSL}(2; \mathbb{R})$ for some cocompact lattice Γ in $\mathrm{PSL}(2; \mathbb{R})$)
- $F_h := \{F_{h,t}\}_{t \in \mathbb{R}}$, horocycle flow on M
- $F_g := \{F_{g,t}\}_{t \in \mathbb{R}}$, geodesic flow on M

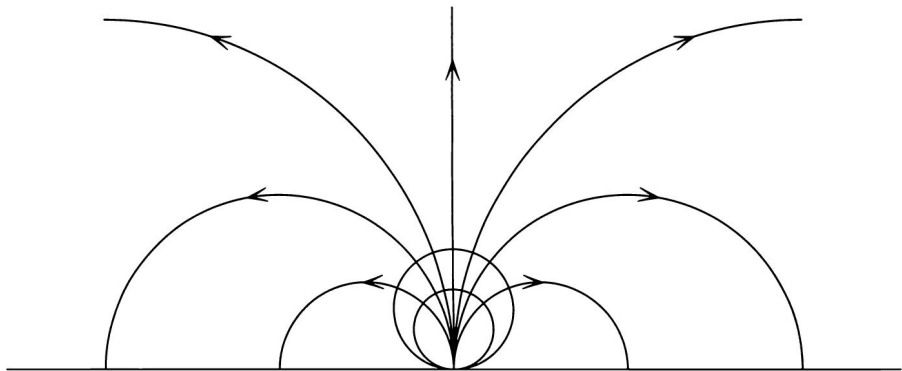
The flows F_h, F_g are one-parameter groups of diffeomorphisms preserving the measure μ .



Geodesic flow in the Poincaré half plane



Positive horocycle flow in the Poincaré half plane
(from Bekka/Mayer's book)



Geodesics and horocycles in the Poincaré half plane
(from Hasselblatt/Katok's book)

Each flow has an essentially self-adjoint generator

$$H_j \varphi := -iX_j \varphi, \quad \varphi \in C^\infty(M) \subset L^2(M, \mu),$$

with X_j the vector field associated to F_j . H_h is of class $C^1(H_g)$ with

$$[iH_h, H_g] = H_h.$$

A C^1 -time change of X_h is a vector field fX_h with $f \in C^1(M; (0, \infty))$. fX_h has a complete flow $\tilde{F}_h := \{\tilde{F}_{h,t}\}_{t \in \mathbb{R}}$ with generator $H := fH_h$ essentially self-adjoint on $C^1(M) \subset \mathcal{H} := L^2(M, f^{-1}\mu)$.

The operator

$$A := f^{1/2} H_g f^{-1/2}$$

is self-adjoint in \mathcal{H} , and $(H - i)^{-1} \in C^1(A)$ with

$$(H + i)^{-1} [iH, A] (H - i)^{-1} = (H + i)^{-1} (H\xi + \xi H) (H - i)^{-1}$$

and

$$\xi := \frac{1}{2} - \frac{1}{2} f^{-1} X_g(f).$$

So,

$$\begin{aligned} D_t &:= \frac{1}{t} \int_0^t ds e^{isH} (H + i)^{-1} [iH, A] (H - i)^{-1} e^{-isH} \\ &= (H + i)^{-1} (H\xi_t + \xi_t H) (H - i)^{-1} \end{aligned}$$

with

$$\xi_t := \frac{1}{t} \int_0^t ds e^{isH} \xi e^{-isH} = \frac{1}{t} \int_0^t ds (\xi \circ \tilde{F}_{h,-s}).$$

Since F_h is uniquely ergodic with respect to μ , \tilde{F}_h is uniquely ergodic with respect to $\tilde{\mu} := \frac{f^{-1}\mu}{\int_M f^{-1}d\mu}$. Thus,

$$\lim_{t \rightarrow \infty} \xi_t = \frac{1}{2} - \frac{1}{2} \int_M d\tilde{\mu} f^{-1} X_g(f) = \frac{1}{2} + \frac{1}{2 \int_M f^{-1} d\mu} \int_M d\mu X_g(f^{-1}) = \frac{1}{2}$$

uniformly on M , and

$$D := \text{s-lim}_{t \rightarrow \infty} D_t = (H + i)^{-1} \left(H \cdot \frac{1}{2} + \frac{1}{2} \cdot H \right) (H - i)^{-1} = H(H^2 + 1)^{-1}.$$

So, $\ker(D) = \ker(H)$, and the theorem implies that

$$\lim_{t \rightarrow \infty} \langle \varphi, e^{-itH} \psi \rangle = 0 \quad \text{for all } \varphi \in \mathcal{H} \text{ and } \psi \in \ker(H)^\perp.$$

Therefore, C^1 -time changes of horocycle flows are strongly mixing.

Thank you !

References

- A. Iwanik, M. Lemańczyk, and C. Mauduit. Absolutely continuous cocycles over irrational rotations. *Israel J. Math.*, 1993
- B. Marcus. Ergodic properties of horocycle flows for surfaces of negative curvature. *Ann. of Math.*, 1977
- R. Tiedra de Aldecoa. Commutator criteria for strong mixing. to appear in *Ergodic Theory Dynam. Systems*
- R. Tiedra de Aldecoa. The absolute continuous spectrum of skew products of compact Lie groups. to appear in *Israel J. Math.*