On one-parameter Koopman groups

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1-10-2015

Outline

Koopman operators

Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space.
Let $T\colon X \to X$ be an (a.e.) invertible, measurable and measure-preserving map.
Define $U_T\colon L_2(X) \to L_2(X)$ by

$$U_T f := f \circ T \quad \text{for all } f \in L_2(X).$$

Name: Koopman operator. Clearly U_T is unitary.

Problem How to recognize that a unitary operator is a Koopman operator? Or unitarily equivalent to a Koopman operator?

Theorem A unitary operator U on $L_2(X)$ is a Koopman operator if and only if

$$U(f\,g) = U(f)\,U(g)$$

for all $f, g \in L_\infty(X)$.

Koopman group

Definition A unitary one-parameter C_0-group $(U_t)_{t\in\mathbb{R}}$ is called a Koopman group if for all $t \in \mathbb{R}$ there exists a measurable $T_t \colon X \to X$ such that

$$U_t f = f \circ T_t \quad \text{for all } f \in L_2(X).$$

Clearly: if $(U_t)_{t\in\mathbb{R}}$ is a Koopman group, then

$$U_t L_\infty(X) \subset L_\infty(X)$$

for all $t \in \mathbb{R}$.

Theorem (Stone) Let A be the generator of a one-parameter C_0-group U. Then U is unitary if and only if A is skew-adjoint.

Problem How to recognize that a skew-adjoint operator is the generator of a Koopman group?

Derivations

Definition Let A be an operator in a function space E and let $\mathcal{D} \subset D(A)$ be an algebra.
Then A is called a derivation on $\mathcal{D}$ if

$$A(f\,g) = (Af)\,g + f\,(Ag) \quad \text{for all } f, g \in \mathcal{D}.$$

Sufficient condition

Theorem (Gallavotti–Pulvirenti, 1976)

Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space.
Let U be a unitary one-parameter C_0-group on $L_2(X)$ with generator A.
Let $\mathcal{D} \subset D(A) \cap L_\infty(X)$.
Suppose that

- $\mathcal{D}$ is a core for A,
- $\mathbb{1} \in \mathcal{D}$,
- $\mathcal{D}$ is an algebra,
- $\mathcal{D}$ is self-adjoint (that is if $f \in \mathcal{D}$ then $\overline{f} \in \mathcal{D}$),
- A is a derivation on $\mathcal{D}$ and
- $\overline{Af} = A\overline{f}$ for all $f \in \mathcal{D}$.

Then for all $t \in \mathbb{R}$ there exists an a.e. invertible measurable and measure preserving map $T_t \colon X \to X$ such that

$$U_t f = f \circ T_t \quad \text{for all } f \in L_2(X).$$

Characterisation

Theorem (tE–Lemańczyk)
Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space.
Let U be a unitary one-parameter C_0-group on $L_2(X)$ with generator A.
Then the following are equivalent.

I. For all $t \in \mathbb{R}$ there exists an a.e. invertible measurable and measure preserving map $T_t \colon X \to X$ such that

$$U_t f = f \circ T_t \quad \text{for all } f \in L_2(X).$$

II.
- The space $L_\infty(X)$ is invariant under U,
- the space $D(A) \cap L_\infty(X)$ is an algebra and
- A is a derivation on $D(A) \cap L_\infty(X)$.

Extension

Unitary is not needed.

Theorem Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space.
Let U be a one-parameter C_0-group on $L_2(X)$ with generator A.
Then the following are equivalent.

I. For all $t \in \mathbb{R}$ there exists a measurable map $T_t \colon X \to X$ such that

$$U_t f = f \circ T_t \quad \text{for all } f \in L_2(X).$$

II.
- The space $L_\infty(X)$ is invariant under U,
- the space $D(A) \cap L_\infty(X)$ is an algebra and
- A is a derivation on $D(A) \cap L_\infty(X)$.

There is also an extension for σ-finite measure spaces.

Weighted non-singular C_0-groups

Theorem Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space.
Let U be a unitary C_0-group on $L_2(X)$ with generator A. Suppose

- U preserves $L_\infty(X)$,
- $\mathbb{1} \in D(A)$ and
- $A\mathbb{1} \in L_\infty(X)$.

Then the following are equivalent.

I. For all $t \in \mathbb{R}$ there exist an a.e. invertible, measurable and measure-preserving map $T_t \colon X \to X$ and a function $\psi_t \colon X \to \mathbb{C}$ such that

$$U_t f = \psi_t \cdot (f \circ T_t) \quad \text{for all } f \in L_2(X).$$

II. For all $t \in \mathbb{R}$ one has $|U_t \mathbb{1}| = 1$ a.e.,
$D(A) \cap L_\infty(X)$ is an algebra and
$A - (A\mathbb{1})I$ is a derivation on $D(A) \cap L_\infty(X)$.

Weighted non-singular C_0-groups

Unitary is not needed.

Theorem Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. Let U be a C_0-group on $L_2(X)$ with generator A. Suppose

- U preserves $L_\infty(X)$,
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- $A\mathbb{1} \in L_\infty(X)$.

Then the following are equivalent.

I. For all $t \in \mathbb{R}$ there exist an a.e. invertible, measurable map $T_t \colon X \to X$ and a function $\psi_t \colon X \to \mathbb{C}$ such that

$$U_t f = \psi_t \cdot (f \circ T_t) \quad \text{for all } f \in L_2(X).$$

II. $D(A) \cap L_\infty(X)$ is an algebra and $A - (A\mathbb{1})I$ is a derivation on $D(A) \cap L_\infty(X)$.

Set-up

Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space.
For all $t \in \mathbb{R}$ let $T_t \colon X \to X$ be a measurable map. Define

$$V_t f := f \circ T_t \quad \text{for all } t \in \mathbb{R}.$$

Assume $V = (V_t)_{t \in \mathbb{R}}$ is a C_0-group on $L_2(X)$.
A cocycle (over V) is a map $\psi \colon \mathbb{R} \to L_\infty(X)$ such that

$$\psi_{t+t'} = \psi_t \cdot (\psi_{t'} \circ T_t)$$

for all $t, t' \in \mathbb{R}$, where $\psi_t = \psi(t)$. Define

$$U_t = \psi_t \, V_t \quad \text{i.e.} \quad U_t f = \psi_t \cdot (f \circ T_t)$$

for all $t \in \mathbb{R}$ and $f \in L_2$.
Clearly $U = (U_t)_{t \in \mathbb{R}}$ is a one-parameter group.

C_0-cocycle

Theorem

The following are equivalent.

I. U is a C_0-group.

II. $\lim_{t\to 0} \|\psi_t - \mathbb{1}\|_2 = 0$.

The main difficulty in the proof of II⇒I is to show that

$$\sup_{t\in(0,1)} \|\psi_t\|_\infty < \infty.$$

Example

Let $\zeta \in L_\infty(X)$. Define $\psi \colon \mathbb{R} \to L_\infty(X)$ by

$$\psi_t := e^{\int_0^t \zeta\circ T_s \, ds}.$$

Then ψ is a cocycle and U is a C_0-group.

Consistent semigroups

Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $p, q \in [1, \infty]$, let U be a one-parameter (semi)group on $L_p(X)$ and let V be a one-parameter (semi)group on $L_q(X)$.
We say that U and V are consistent if

$$U_t f = V_t f$$

for all $t \in (0, 1)$ and $f \in L_p(X) \cap L_q(X)$.

Problem Let $p \in [1, \infty)$. Let S be a C_0-(semi)group on $L_2(X)$ which extends consitently to a (semi)group V on $L_p(X)$.
If V then also a C_0-(semi)group?

Solution: Suppose in addition that $\sup_{t \in (0,1)} \|V_t\|_{p \to p} < \infty$.

Then yes if $p > 1$.
There are a few sufficient conditions if $p = 1$.

Consistent semigroups

Theorem (tE–Lemańczyk)
Let $(X, \mathcal{B}, \mu)$ be a finite measure space.
Let S be a C_0-group on $L_2(X)$.
Then the following are equivalent.

I. The group S extends consistently to a C_0-group on $L_1(X)$.
II. The space $L_\infty(X)$ is invariant under S^*.
(Thus $S^*_t(L_\infty(X)) \subset L_\infty(X)$ for all $t \in \mathbb{R}$.)

If the (equivalent) conditions are valid, then there exist $M \geq 1$ and $\omega \geq 0$ such that

$$\|S^*_t f\|_\infty \leq M \, e^{\omega |t|} \, \|f\|_\infty$$

for all $t \in \mathbb{R}$ and $f \in L_\infty(X)$.

Proof of II⇒I

Closed graph theorem: $\forall_{t\in\mathbb{R}}\exists_{c>0}\forall_{f\in L_\infty}\ \|S_t^* f\|_\infty \leq c\,\|f\|_\infty$.

Hence there are group $\widehat{S}$ on L_1 consistent with S
group $\widetilde{S}$ on L_∞ consistent with S^*.
Moreover, $\widetilde{S}_t = (\widehat{S}_t)^*$ for all $t \in \mathbb{R}$.

Main difficulty: is $\{\widetilde{S}_t : t \in [2,3]\}$ bounded in $\mathcal{B}(L_\infty)$?
Claim: $\{\|\widetilde{S}_t f\|_\infty : t \in [2,3]\}$ is bounded for all $f \in L_\infty$.

For the proof of the claim use arguments as in [ABHN] Lemma 3.16.4.

Proof of II⇒I

Recall $\widetilde{S}$ is group on L_∞ which is consistent with S^*.

Fix $f \in L_\infty$. If $t \in \mathbb{R}$ then

$$\begin{aligned} \|\widetilde{S}_t f\|_\infty &= \sup\{|\langle \widetilde{S}_t f, g\rangle| : g \in L_1 \text{ and } \|g\|_1 \le 1\} \\ &= \sup\{|\langle \widetilde{S}_t f, g\rangle| : g \in L_2 \text{ and } \|g\|_1 \le 1\} \\ &= \sup\{|(f, S_t g)| : g \in L_2 \text{ and } \|g\|_1 \le 1\}. \end{aligned}$$

For each $g \in L_2$ the map $t \mapsto |(f, S_t g)|$ is continuous by the strong continuity of S on L_2.

So $t \mapsto \|\widetilde{S}_t f\|_\infty$ is lower semicontinuous, hence measurable function on $\mathbb{R}$.

Proof of II⇒I

Recall $\widetilde{S}$ is group on L_∞ which is consistent with S^*.

If $f \in L_\infty$, then $t \mapsto \|\widetilde{S}_t f\|_\infty$ is a measurable function on $\mathbb{R}$.

Fix $f \in L_\infty$.
Suppose that $\{\|\widetilde{S}_t f\|_\infty : t \in [2,3]\}$ is not bounded.
For all $n \in \mathbb{N}$ choose $t_n \in [2,3]$ with $\|\widetilde{S}_{t_n} f\|_\infty \geq n$.
Wlog: $t_n \to t_0 \in [2,3]$.
Since $t \mapsto \|\widetilde{S}_t f\|_\infty$ is measurable, there are $M > 0$ and a measurable set $F \subset [0, t_0]$ such that $\lambda(F) > 1$ and $\|\widetilde{S}_t f\|_\infty \leq M$ for all $t \in F$.
Let $n \in \mathbb{N}$. Then

$$n \leq \|\widetilde{S}_{t_n} f\|_\infty \leq \|\widetilde{S}_{t_n - t}\| \, \|\widetilde{S}_t f\|_\infty \leq M \, \|\widetilde{S}_{t_n - t}\|$$

for all $t \in F$.
So $\|\widetilde{S}_s\| \geq M^{-1} \, n$ for all $s \in E_n$, where

$$E_n = \{t_n - t : t \in F \cap [0, t_n]\}.$$

Proof of II⇒I

Recall $\|\widetilde{S}_s\| \geq M^{-1}\,n$ for all $s \in E_n$, where $E_n = \{t_n - t : t \in F \cap [0, t_n]\}$.
Also $t_n \to t_0 \in [2, 3]$.
Measurable $F \subset [0, t_0]$ such that $\lambda(F) > 1$ and $\|\widetilde{S}_t f\|_\infty \leq M$ for all $t \in F$.

Note that E_n is measurable and $\lambda(E_n) \geq 1$ if $|t_n - t_0| < \lambda(F) - 1$.

Let $E = \limsup_{n\to\infty} E_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n$.
Then E is measurable and $\lambda(E) \geq 1$.
In particular, $E \neq \emptyset$.
Moreover, $\|\widetilde{S}_s\| = \infty$ for all $s \in E$. Contradiction!

Uniform boundedness principle: $\{\widetilde{S}_t : t \in [2, 3]\}$ is bounded in $\mathcal{B}(L_\infty)$

Proof of II⇒I

Recall group $\widehat{S}$ on L_1 consistent with S
group $\widetilde{S}$ on L_∞ consistent with S^*.
$\widetilde{S}_t = (\widehat{S}_t)^*$ for all $t \in \mathbb{R}$.

Conclusion: $\{\widehat{S}_t : t \in [2,3]\}$ is bounded in $\mathcal{B}(L_1)$.
Group property: $\{\widehat{S}_t : t \in [-1,1]\}$ is also bounded in $\mathcal{B}(L_1)$.
Let $c = \sup\{\|\widehat{S}_t\| : t \in [-1,1]\} < \infty$.
Let $g \in L_\infty$. Then

$$\lim_{t \to 0} \langle g, \widehat{S}_t f \rangle = \lim_{t \to 0} (g, S_t f) = (g, f) = \langle g, f \rangle \quad \text{for all } f \in L_2.$$

Since L_2 is dense in L_1 and $c < \infty$ also

$$\lim_{t \to 0} \langle g, \widehat{S}_t f \rangle = \langle g, f \rangle \quad \text{for all } f \in L_1.$$

So $\widehat{S}$ is weakly continuous and hence $\widehat{S}$ is a C_0-group. □

Reference

- A. F. M. ter Elst and M. Lemańczyk,
  On one-parameter Koopman groups.
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