Asymptotics of infinite systems of ODEs

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Frontiers of Operator Dynamics

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Overview

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Example 1: The ‘robot rendezvous problem’

Consider countably many robots at positions $x_k(t) \in \mathbb{C}$, where $t \geq 0$ and $k \in \mathbb{Z}$. Aim for mutual ‘rendezvous’ by setting $\dot{x}_k(t) = u_k(t)$, where

$$u_k(t) = x_{k-1}(t) - x_k(t).$$

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Cauchy problem:

$$\begin{cases} \dot{x}(t) = Ax(t), & t \geq 0, \\ x(0) = x_0 \in X, \end{cases} \tag{CP-Rob}$$

where $x(t) = (x_k(t))_{k\in\mathbb{Z}}$, $A = S - I$ for $S =$ right-shift on $X = \ell^p(\mathbb{Z})$, $1 \leq p \leq \infty$.

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Theorem (Feintuch, Francis ’12)

For $p = \infty$ the solution $x(t)$, $t \geq 0$, of (CP-Rob) need not converge to a limit as $t \to \infty$, but we always have $\dot{x}(t) \to 0$.

Example 2: The platoon system

Consider a more realistic model in which vehicle $k \in \mathbb{Z}$ at time $t \geq 0$ has position $s_k(t)$, velocity $v_k(t)$ and acceleration $a_k(t)$.

Aim to steer vehicle k towards *target separation* q_k from vehicle $k-1$ and have all vehicles moving at *target velocity* v, by controlling its acceleration: $\dot{a}_k(t) = u_k(t)$.

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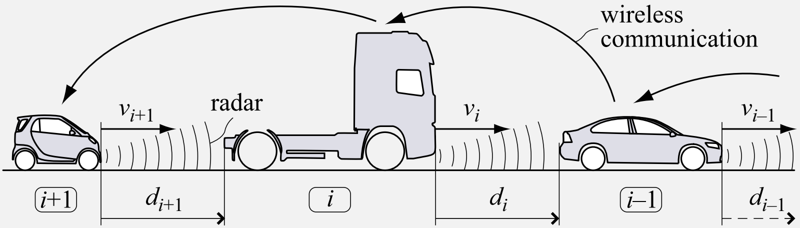
The state vector of vehicle k is

$$x_k(t) = \begin{pmatrix} y_k(t) \\ w_k(t) \\ a_k(t) \end{pmatrix} = \begin{pmatrix} q_k - d_k(t) \\ v_k(t) - v \\ a_k(t) \end{pmatrix},$$

where $d_k(t) = s_{k-1}(t) - s_k(t)$.

A picture



Taken from Ploeg, van de Wouw, Nijmeijer '14

Equations of motion

We choose the 'state feedback control'

$$u_k(t) = -\alpha_0 y_k(t) - \alpha_1 w_k(t) - \alpha_2 a_k(t).$$

Then

$$\dot{x}_k(t) = \begin{pmatrix} w_k(t) - w_{k-1}(t) \\ a_k(t) \\ u_k(t) \end{pmatrix} = A_0 x_k(t) + A_1 x_{k-1}(t),$$

where

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{pmatrix} \qquad A_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Cauchy problem

Letting $x(t) = (x_k(t))_{k\in\mathbb{Z}}$ we write this as

$$\begin{cases} \dot{x}(t) = Ax(t), & t \geq 0, \\ x(0) = x_0 \in X, \end{cases} \qquad \text{(CP-Plat)}$$

where $X = \ell^p(\mathbb{Z};\mathbb{C}^3)$, $1 \leq p \leq \infty$, and

$$A = \begin{pmatrix} \ddots & & & & \\ \ddots & A_0 & & & \\ & A_1 & A_0 & & \\ & & A_1 & A_0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

Asymptotics of the platoon system

Theorem (Zwart?)

For $p = 2$ and suitable choices of $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{C}$ we get $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in X$.

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Questions

- Do solutions converge for $p \neq 2$?
- If not for all x_0 then for which ones?
- When we have convergence, is there a rate?
- Is it still true that $\dot{x}(t) \to 0$ as $t \to \infty$?
- If so, how fast?

The general Cauchy problem

We consider the more general problem

$$\begin{cases} \dot{x}(t) = Ax(t), & t \geq 0, \\ x(0) = x_0 \in X, \end{cases} \tag{CP}$$

where $X = \ell^p(\mathbb{Z}; \mathbb{C}^m)$ with $1 \leq p \leq \infty$, $m \in \mathbb{N}$ and where

$$A = \begin{pmatrix} \ddots & & & & \\ \ddots & A_0 & & & \\ & A_1 & A_0 & & \\ & & A_1 & A_0 & \\ & & & \ddots & \ddots \end{pmatrix}$$

for suitable matrices $A_0, A_1 \in \mathbb{C}^{m \times m}$.

Overview

The semigroup approach

Observe that the solution of (CP) is given, for $t \geq 0$, by

$$x(t) = T(t)x_0,$$

where

$$T(t) = \exp(tA) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k.$$

In particular,

$$\dot{x}(t) = AT(t)x_0 = T(t)Ax_0.$$

Objective

To understand the asymptotic behaviour of solutions to (CP) by studying the semigroup T and its generator A.

An abstract result

Notation: For $\lambda \notin \sigma(A)$, we let $R(\lambda, A) = (\lambda - A)^{-1}$.

Theorem (Batty, Chill, Tomilov '14; Chill, S '15)

Let T be a bounded semigroup whose generator $A \in \mathcal{B}(X)$ satisfies $\sigma(A) \cap i\mathbb{R} = \{0\}$. Suppose further that

$$\|R(is, A)\| = O\big(|s|^{-\alpha}\big), \quad s \to 0,$$

for some $\alpha \geq 1$. Then

$$\|AT(t)\| = O\left(\left(\frac{\log t}{t}\right)^{1/\alpha}\right), \quad t \to \infty.$$

If X is a Hilbert space, the logarithmic term can be dropped.

Two assumptions

Assumption (A1)

We have $A_1 \neq 0$.

Assumption (A2)

There exists a function ϕ such that

$$A_1 R(\lambda, A_0) A_1 = \phi(\lambda) A_1, \quad \lambda \notin \sigma(A_0).$$

We call ϕ the *characteristic function*.

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We call ϕ the *characteristic function*.

- holds automatically if $\operatorname{rank} A_1 = 1$
- ϕ is rational with poles contained in $\sigma(A_0)$
- $|\phi(\lambda)| \to 0$ as $|\lambda| \to \infty$

The spectrum of A

Theorem (Paunonen, S '15)

Let $1 \leq p \leq \infty$ and $m \in \mathbb{N}$, and suppose that (A1), (A2) hold. Let

$$\Omega_\phi = \big\{\lambda \in \mathbb{C}\backslash\sigma(A_0) : |\phi(\lambda)| = 1\big\}.$$

Then $\Omega_\phi = \sigma(A)\backslash\sigma(A_0)$ and given $\lambda \in \Omega_\phi$ we have

- $\lambda \in \sigma_p(A)$ *if and only if* $p = \infty$,
- $\lambda \in \sigma_r(A)$ *if and only if* $p = 1$ *or* $p = \infty$.

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- *$\lambda \in \sigma_p(A)$ if and only if $p = \infty$,*
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Also know that

- if $p = \infty$ and $\lambda \in \Omega_\phi$ then $\dim \operatorname{Ker}(\lambda - A) = \operatorname{rank} A_1$
- points in $\sigma(A_0)$ can lie inside or outside $\sigma(A)$

Growth of the resolvent

Theorem (Paunonen, S '15)

Let $1 \leq p \leq \infty$ *and* $m \in \mathbb{N}$, *and suppose that (A1), (A2) hold. If* $\mu \in \Omega_\phi$ *then*

$$\|R(\lambda, A)\| \asymp \frac{1}{|1 - |\phi(\lambda)||}$$

as $\lambda \to \mu$ *with* $\lambda \notin \sigma(A)$.

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Assumption (A3)

We have $\sigma(A_0) \subset \mathbb{C}_- = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\}$.

Assumption (A4)

We have $0 \in \Omega_\phi \subset \mathbb{C}_- \cup \{0\}$ and $\phi'(0) \neq 0$.

A sufficient condition for boundedness

Theorem (Paunonen, S '15)

Let $1 \le p \le \infty$ and $m \in \mathbb{N}$, and suppose that (A1)–(A4) hold. Let $Q = \{\lambda \in \mathbb{C} : 0 < \operatorname{Re}\lambda \le 1, |\operatorname{Im}\lambda| \le \|A\| + 1\}$. The semigroup T generated by A is bounded provided

$$\sup_{\lambda \in Q} \sup_{k \ge 0} \frac{(\operatorname{Re}\lambda)^{k+1}}{k!} \sum_{j=0}^{\infty} |D^k \phi(\lambda)^j| < \infty. \qquad (*)$$

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Assumption (A5)

The function ϕ satisfies $(*)$.

Towards an asymptotic result

Would like to characterise the set

$$C = \left\{ x_0 \in X : \lim_{t \to \infty} x(t) \text{ exists} \right\},$$

where $x(t)$, $t \geq 0$, is the solution of (CP) with initial data x_0.

Also hope to describe the limit when $x_0 \in C$, to show that $\dot{x}(t) \to 0$ for all $x_0 \in X$, and obtain rates where possible.

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Note: From (A1)–(A4) we have $\sigma(A) \cap i\mathbb{R} = \{0\}$ and

$$\|R(is, A)\| \asymp |s|^{-n}, \quad s \to 0,$$

for some $n = n_\phi \in 2\mathbb{N}$.

Convergence of solutions

Let $1 \le p \le \infty$, $m \in \mathbb{N}$ and suppose that (A1)–(A5) hold.

Notation: Let $M \in \mathcal{B}(X)$ be given by $M(x_k) = (A_1 A_0^{-1} x_k)$.

Theorem (Paunonen, S '15)

Given $x_0 \in X$ *we have* $x_0 \in C$ *if and only if*

$$\left\| \frac{1}{n} \sum_{k=1}^{n} \phi(0)^k S^k M x_0 - y \right\| \to 0, \quad n \to \infty, \qquad (\diamondsuit)$$

for some $y = (\phi(0)^k y_0)$ *with* $y_0 \in \operatorname{Ran} A_1$.

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for some $y = (\phi(0)^k y_0)$ with $y_0 \in \operatorname{Ran} A_1$. Moreover, there exists a matrix L such that if $(\diamondsuit)$ holds, then for $z = (\phi(0)^k L y_0)$ we have

$$\|x(t) - z\| \to 0, \quad t \to \infty.$$

In particular, $C = X$ if and only if $1 < p < \infty$.

Rates of convergence

Theorem (Paunonen, S '15)

If $x_0 \in C$ *and the convergence in* $(\diamondsuit)$ *is like* $O(n^{-1})$ *as* $n \to \infty$, *then*

$$\|x(t) - z\| = O\left(\left(\frac{\log t}{t}\right)^{1/n_\phi}\right), \quad t \to \infty.$$

Moreover, for all $x_0 \in X$ *we have*

$$\|\dot{x}(t)\| = O\left(\left(\frac{\log t}{t}\right)^{1/n_\phi}\right), \quad t \to \infty.$$

In both cases the logarithm can be dropped if $p = 2$.

Overview

The robot rendezvous problem

Here $m = 1$, $A_0 = -1$ and $A_1 = 1$. So (A1)–(A5) hold with

$$\phi(\lambda) = \frac{1}{\lambda + 1} \quad \text{and} \quad n_\phi = 2.$$

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Corollary (Paunonen, S '15)

Let $1 \leq p \leq \infty$. Given $x_0 \in X$ we have $x_0 \in C$ if and only if

$$\left\| \frac{1}{n} \sum_{k=1}^{n} S^k x_0 - y \right\| \to 0, \quad n \to \infty, \tag{$\sharp$}$$

for some constant sequence $y \in X$. If ($\sharp$) holds then $x(t) \to y$.

In particular, $C = X$ if and only if $1 < p < \infty$.

Rates of convergence

Proposition (Paunonen, S '15)

Let $1 \le p \le \infty$. *If* $x_0 \in C$ *and*

$$\left\| \frac{1}{n} \sum_{k=1}^{n} S^k x_0 - y \right\| = O(n^{-1}), \quad n \to \infty,$$

then

$$\|x(t) - y\| = O(t^{-1/2}), \quad t \to \infty.$$

Moreover, for all $x_0 \in X$ *we have*

$$\|\dot{x}(t)\| = O(t^{-1/2}), \quad t \to \infty.$$

The platoon model

Now $m = 3$ and

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{pmatrix} \qquad A_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So (A1), (A2) hold with

$$\phi(\lambda) = \frac{\alpha_0}{p(\lambda)},$$

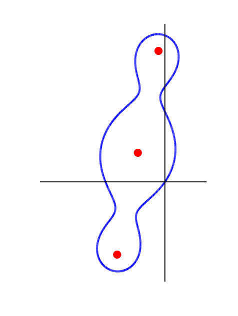
where

$$p(\lambda) = \lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$$

is the characteristic polynomial of A_0.

Placing the poles

Possible choices of $\sigma(A_0)$ and the resulting Ω_ϕ:



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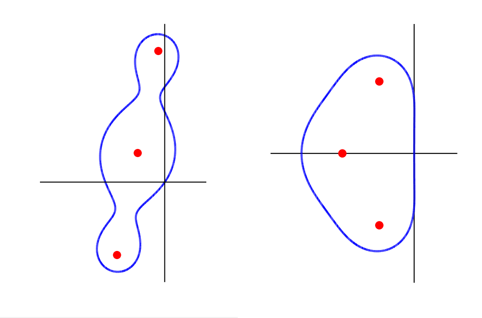

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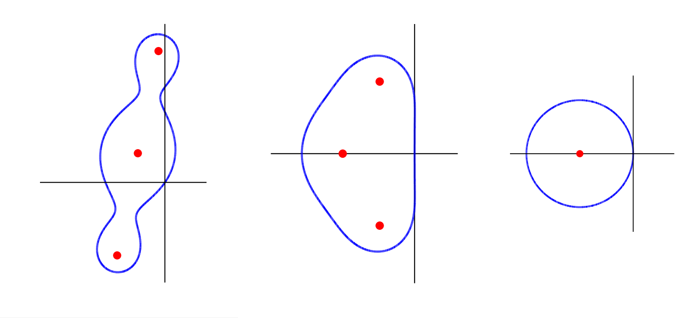

Choose $\alpha_0 = 1$, $\alpha_1 = \alpha_2 = 3$, so that $p(\lambda) = (\lambda + 1)^3$.

Then (A1)–(A5) hold and $n_\phi = 2$.

Convergence of solutions

Corollary (Paunonen, S '15)

Let $1 \leq p \leq \infty$ and let α_0, α_1, α_2 be as above. Given $x_0 \in X$ we have $x_0 \in C$ if and only if there exists $y \in \ell^p(\mathbb{Z})$ such that

$$\left\| \frac{1}{n} \sum_{k=1}^{n} S^k y_0 - y \right\|_{\ell^p(\mathbb{Z})} \to 0, \quad n \to \infty, \tag{$\dagger$}$$

where y_0 is the vector of initial deviations.

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where y_0 is the vector of initial deviations. If $(\dagger)$ holds then $y = (\dots, c, c, c, \dots)$ for some $c \in \mathbb{C}$ and $x(t) \to z$ where

$$z = \left(\dots, \begin{pmatrix} c \\ -c/3 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ -c/3 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ -c/3 \\ 0 \end{pmatrix}, \dots \right).$$

In particular, $C = X$ if and only if $1 < p < \infty$.

Rates of convergence

Corollary (Paunonen, S '15)

Let $1 \le p \le \infty$ and let α_0, α_1, α_2 be as before. If $x_0 \in C$ and if the convergence in (†) is like $O(n^{-1})$ as $n \to \infty$, then

$$\|x(t) - z\| = O\left(\left(\frac{\log t}{t} \right)^{1/2} \right), \quad t \to \infty.$$

Moreover, for any $x_0 \in X$ we have

$$\|\dot{x}(t)\| = O\left(\left(\frac{\log t}{t} \right)^{1/2} \right), \quad t \to \infty.$$

In both cases the logarithm can be dropped if $p = 2$.

Thank you.