Automorphism Groups of Minimal Subshifts of low complexity

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Basic topological notions

Definition

Let (X, T) be a topological dynamical system, X a topological space. An automorphism $\phi\colon X \to X$ is an homeomorphism s.t.

$$\phi \circ T = T \circ \phi.$$

$$\mathrm{Aut}(X, T) = \{\phi \text{ automorphism of } (X, T)\}.$$

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Q : What can we say on $\mathrm{Aut}(X, T)$?

Let A be a finite alphabet.
$A^{\mathbb{Z}}$ endowed with the product topology.
The shift map

$$\begin{aligned} \sigma : A^{\mathbb{Z}} &\to A^{\mathbb{Z}} \\ (x_n)_{n\in\mathbb{Z}} &\mapsto (x_{n+1})_{n\in\mathbb{Z}} \end{aligned}$$

For a closed set $X \subset A^{\mathbb{Z}}$, shift invariant ($\sigma(X) = X$), a subshift is the dynamical system $(X, \sigma_{|X})$.

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Theorem (Curtis-Hedlund-Lyndon)

Let ϕ be an automorphism of (X, σ)
There exists a local map $\hat{\phi}$: $A^{2r+1} \to A$ *s.t.*

$$\phi(x)_n = \hat{\phi}(x_{n-r} \cdots x_{n+r}) \textit{ for any } n \in \mathbb{Z}.$$

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e.g. $A = \{0, 1\}$, $\hat{\phi}$:

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Corollary

$Aut(X, \sigma)$ is countable.
$Aut(X, \sigma)$ is a discrete subgroup of Homeo(X) for the uniform convergence topology.

Main theorem

(X, σ) is minimal if any orbit is dense in X.

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Let (X, σ) be a minimal subshift. If

$$\liminf_n \frac{p_X(n)}{n} < +\infty,$$

then $Aut(X, \sigma)/\langle \sigma \rangle$ is finite.

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- Primitive substitutive subshifts:
  minimal subshift (X, σ) with a clopen proper subset $U \subset X$
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- Coding of minimal Interval Exchange Transformations.

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Example. Primitive substitutive subshifts:
Generalizes results of V. Salo-I. Törmä.
Similar result by V. Cyr-B. Kra

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Example. This includes also

- Subshifts of polynomial complexity of arbitrarily high degree.
- Subshifts with subexponential complexity $p_X(n) \geq g(n)$ i.o. where $\lim_n g(n)/\alpha^n = 0$ for any $\alpha > 1$.

Centralizer group: for a measurable dynamical system $(X, \mathcal{B}, \mu, T)$,

$$C(T) = \{\phi\colon X \to X;\ \text{bi-measurable},\ \phi_*\mu = \mu,\ \phi \circ T = T \circ \phi\}$$

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- J. King, J.-P. Thouvenot (91): mixing system of finite rank

$$C(T)/\langle T \rangle \text{ is finite.}$$

From the measurable to the topological setting

For non-weakly mixing system:

- B. Host, F. Parreau (89): for a family of substitutive systems

$$C(\sigma) = \mathrm{Aut}(X, \sigma) \text{ and } C(\sigma)/\langle \sigma \rangle \text{ is finite.}$$

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- M. Lemańczyk, M. Mentzen (89): any finite group can be realized as $C(\sigma)/\langle\sigma\rangle$.

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In this case:

$\mathrm{Aut}(X, \sigma)$ is not finitely generated, not amenable.

Zoologie, positive entropy

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Notice this example is not weakly-mixing.
Given by a Toeplitz sequence:
i.e. a subshift $\overline{\{\sigma^n(x) : n \in \mathbb{Z}\}}$ s.t.
for any neighborhood U of x

$$\{n \in \mathbb{Z} : \sigma^n(x) \in U\} \text{ contains a subgroup of } \mathbb{Z}.$$

Main Ideas

Lemma

Let (X, T) be a minimal aperiodic dynamical system. The action of $Aut(X, T)$ on X

$$\begin{array}{rcl} Aut(X,T) \times X & \to & X \\ (\phi, x) & \mapsto & \phi(x), \end{array}$$

is free (the stabilizer of any point is trivial).

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Lemma

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Proof. For any automorphism ϕ, the set

$$\{x; \phi(x) = x\}$$

is closed and T invariant.

Main Ideas

Two orbits $\mathrm{Orb}_T^{(1)}$ and $\mathrm{Orb}_T^{(2)}$ are asymptotic, if they contains asymptotic points, *i.e.*:
$\exists x \in \mathrm{Orb}_T^{(1)}$, $y \in \mathrm{Orb}_T^{(2)}$ s.t.

$$\lim_{n \to +\infty} \mathrm{dist}(T^n(x), T^n(y)) = 0.$$

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- This defines an equivalence relation.
- Any automorphism ϕ maps asymptotic orbits to asymptotic orbits.
- Any automorphism ϕ induces a permutation on the collection of asymptotic class of orbits.

Main Ideas

Corollary

For a minimal t.d.s. (X, T), with two asymptotic orbits, we have

$$\{1\} \longrightarrow \langle T \rangle \longrightarrow \ Aut(X, T) \ \stackrel{j}{\longrightarrow} \ Per\ \mathcal{O},$$

where :

- *$\mathcal{O}$ denote the collection of non trivial asymptotic class of orbits.*
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$j(\phi)$ has a fixed point $\Leftrightarrow \phi \in \langle T \rangle$

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If $\sharp\mathcal{O} = 1$, then $Aut(X,T) = \langle T \rangle$.
e.g. for Sturmian sequences

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If $\sharp\mathcal{O} < +\infty$, then $\sharp Aut(X, T)/\langle T \rangle$ divides $\sharp\mathcal{O}$.

Main Ideas

Proposition

Let (X, σ) be a subshift with $\liminf_n p_X(n)/n < \infty$. Then there is a finite number of asymptotic pair, i.e.

$$\sharp\mathcal{O} < +\infty.$$

More generally

In the same way: $x, y \in X$ are proximal if

$$\liminf_n dist(T^n x, T^n y) = 0.$$

$\phi \in \mathrm{Aut}(X, T)$ maps proximal points to proximal points.

Commutator in a group G: $[g, h] = ghg^{-1}h^{-1}$

$$G_0 = G, \qquad G_j = [G_{j-1}, G] = \langle [a, b];\ a \in G_{j-1}, b \in G \rangle.$$

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A group G is d-step nilpotent if $G_d = \{e\}$.

Example. If $d = 1$, G is Abelian.

G a d-step nilpotent Lie group. $\Gamma \subset G$ a lattice.
Any minimal translation L_g in G/Γ is a nil translation.

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Theorem (DDMP)

If $\pi\colon (X,T) \to \varprojlim_i (G_i/\Gamma_i, L_{g_i})$ is a proximal extension of an inverse limit of minimal d-nil translation,
then $Aut(X,T)$ is a d-step nilpotent group.
Moreover, $\hat{\pi}\colon Aut(X,T) \to Aut(\varprojlim_i (G_i/\Gamma_i, L_{g_i}))$ is injective.

Theorem (DDMP)

If (X, T) is a minimal proximal extension of its maximal non trivial d-step nilfactor (X_d, T_d). Then $Aut(X, T)$ embeds into $Aut(X_d, T_d)$, and $Aut(X, T)$ is a d-step nilpotent group.

Example. Toeplitz subshifts are proximal extension of their maximal equicontinuous factor ($d = 1$).

Their automorphism group is Abelian.

Open questions

Question

Given a countable group G. Does it exists a minimal subshift such that $\mathrm{Aut}(X, \sigma)/\langle \sigma \rangle$ is isomorphic to G ?

Open questions

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Given a countable group G. Does it exists a minimal subshift such that $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ is isomorphic to G ?

True for G: finite, $\mathbb{Z}^d$
Salo: example $\mathrm{Aut}(X,\sigma)$ is Abelian not finitely generated

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Relation between growth rate of $\mathrm{Aut}(X,\sigma)$ and the complexity ?

Cyr and Kra: if $p_X(n)/n^2 \to 0$ then $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ is periodic.