Some remarks
regarding ergodic operators

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(joint with S. Grivaux)

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Basic question. How can we *see*

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Basic question. How can we *see* that an operator is or is not ergodic?

1. Ergodicity and frequent hypercyclicity

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2. Ergodicity and unimodular eigenvectors

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Interlude : perfect spanning vs “chaos”

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3. A useful parameter

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- Obviously, if T is frequently hypercyclic, then $c(T) > 0$.

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so that $\underline{\mathrm{dens}}\, \mathcal{N}_T(x, V) = 0$. Hence, no $x \in G$ can be a frequently hypercyclic vector for T. In particular: *the set of all frequently hypercyclic vectors for T is always meager in X.*

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$$c(T) \geq \overline{\mathrm{dens}}\, \mathcal{N}_T(x, B_2) \geq \underbrace{\overline{\mathrm{dens}}\, \mathcal{N}_T(x, B_1)}_{c(T)} + \underline{\mathrm{dens}}\, \mathcal{N}_T(x, V)\,,$$

so that $\underline{\mathrm{dens}}\, \mathcal{N}_T(x, V) = 0$. Hence, no $x \in G$ can be a frequently hypercyclic vector for T. In particular: *the set of all frequently hypercyclic vectors for T is always meager in X.* (Moothathu 2013, Bayart-Ruzsa 2014, ...)

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