Some Important Theorems

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The topological case

Theorem 1. K, L compact, $T : \mathrm{C}(K) \to \mathrm{C}(L)$ linear, $T\mathbf{1} = \mathbf{1}$.
Then the following assertions are equivalent:

(i) T multiplicative;

(ii) T lattice homomorphism;

(iii) T extreme point of $\{S : \mathrm{C}(K) \to \mathrm{C}(L) \,|\, S \geq 0,\ S\mathbf{1} = \mathbf{1}\}$;

(iv) $T = T_\varphi$ for some $\varphi : L \to K$.

In this case, φ is unique.

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Von Neumann's theorem

Theorem 2. X, Y standard, $T : \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^1(\mathrm{Y})$ Markov
Then the following assertions are equivalent:

(i) T multiplicative on $\mathrm{L}^\infty(\mathrm{X})$;

(ii) T lattice homomorphism, i.e., Markov embedding;

(iii) $T = T_\varphi$ for some measurable $\varphi : Y \to X$.

In this case, φ is essentially unique.

For a (nice!) proof see [EFHN, Chapter 13 and Appendix F].

Topological Models

Theorem 3. $(\mathrm{L}^1(\mathrm{X}); T)$ abstract mps, $\mathbf{1} \in A \subseteq \mathrm{L}^\infty(\mathrm{X})$ a T-invariant C^*-subalgebra.

$\Rightarrow$ There exist

1) a topological system $(K; \varphi)$,
2) a φ-invariant probability measure μ with full support,
3) a Markov embedding $\Phi : \mathrm{L}^1(K, \mu) \to \mathrm{L}^1(\mathrm{X})$,

such that $\Phi T_\varphi = T\Phi$ and $\Phi(\mathrm{C}(K)) = A$.

Proof: Gelfand-Naimark, see [EFHN, Chapter 12].

NB: $\mathrm{ran}(\Phi) = \mathrm{cl}_{\mathrm{L}^1}(A)$, so Φ is an isomorphism iff A dense in $\mathrm{L}^1(\mathrm{X})$.

Abramov's theorem

Theorem 4 (Representation). $\mathbf{X} = (\mathrm{L}^1(\mathrm{X}); T)$ totally ergodic of quasi-discrete spectrum, signature $(H; \Lambda, \eta_1)$. Then $\mathbf{X}$ is isomorphic to the affine automorphism system $(H^*, \mathrm{m}; \Phi^*, \eta)$, where:

1) H^* is the (compact) dual group of $H = H_d$;
2) $\Phi(h) = h\Lambda h$ for $h \in H$;
3) $\eta \in H^*$ is (any!) extension to H of $\eta_1 : H_1 \to \mathbb{T}$.

Recall: the dynamics on $(H^*, \mathrm{m}; \Phi^*, \eta)$ is

$$\chi \mapsto \Phi^*(\chi)\eta = \chi(\chi \circ \Lambda)\eta.$$

Special case (Halmos-von Neumann): $H = H_1$, group rotation.

Abramov's theorem II

Corollary 5 (Isomorphism). Two totally ergodic systems with quasi-discrete spectrum are isomorphic iff their signatures are (in the obvious sense).

Theorem 6 (Realization). Let (H, Λ, η_1) be a signature such that H is torsion-free, and let $\eta \in H^*$ be any extension of η_1 to H. Then the associated affine automorphism system $(H^*, \mathrm{m}; \Phi^*, \eta)$ is totally ergodic and has quasi-discrete spectrum isomorphic to (H, Λ, η_1).

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