Birkhoff and Oseledets genericity along curves

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Birkhoff genericity

Let (X, d) be a locally compact metric space and μ a probability borel measure on X. Let $(\psi_t)_{t\in\mathbb{R}}$ be a measure-preserving flow, i.e. $(\psi_t)_{t\in\mathbb{R}}$ is a family of homeomorphisms of X satisfying:

$$\psi_{t_1+t_2}(x) = \psi_{t_1}(\psi_{t_2}(x)) \text{ and } \psi_0(x) = x.$$

If $(\psi_t)_{t\in\mathbb{R}}$ is ergodic then by Birkhoff's Ergodic Theorem for almost every $x \in X$ and for every compactly supported continuous function $f : X \to \mathbb{R}$ ($f \in C_c(X)$) we have

$$\frac{1}{T}\int_0^T f(\psi_t x)\, dt \to \int_X f\, d\mu \text{ as } T \to \infty.$$

Each such point $x \in X$ is called **Birkhoff generic**.
General problem: to understand the structure of the set of Birkhoff generic point deeper. More precisely, to find curves $\gamma : I \to X$ such that the points $\gamma(s)$ are Birkhoff generic for a.e. $s \in I$.

Oseledets genericity

Suppose that $A : \mathbb{R} \times X \to GL_d(\mathbb{R})$ be a borel cocycle for the flow $(\psi_t)_{t\in\mathbb{R}}$, i.e.

$$A_{t_1+t_2}(x) = A_{t_1}(\psi_{t_2}x) \circ A_{t_2}(x).$$

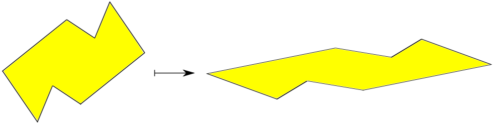
Assume that $x \mapsto \sup_{|t|\leqslant 1} \log \max\{\|A_t(x)\|, \|A_t^{-1}(x)\|\}$ is integrable. Then, by Oseledets Multiplicative Ergodic Theorem (the version proved by Gol'dsheid and Margulis), for almost every $x \in X$

$$(A_t^*(x) \circ A_t(x))^{\frac{1}{2t}} \to \Lambda(t,x) \text{ as } t \to \infty. \tag{1}$$

Essentially, this yields the standard version of Oseledets Theorem with invariant splitting. Every point $x \in X$ for which (1) holds is called **Oseledets generic**.

Problem: to describe curves $\gamma : I \to X$ such that the points $\gamma(s)$ are Oseledets generic for a.e. $s \in I$.

Roughly speaking, a **compact translation surface** is the resulting surface after gluing of the corresponding sides in any compact polygon for which sides are partitioned into pairs of parallel sides of the same length, pair are glued. Then the surface inherits its translation structure form the plane with vertical direction distinguished.



Let us pass to the translation surfaces (M, ω) with fixed genus $g \geqslant 2$. By the moduli space $\mathcal{M}_g$ of translation surfaces of genus g we can mean the quotient space of such polygons when we identify two polygons generating the same translation surface. On $\mathcal{M}_g$ there is a natural action of $SL_2(\mathbb{R})$ defined by linear transformations of polygons. Let us consider two important subactions: the **Teichmüller flow** $a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, $t \in \mathbb{R}$ and the rotation $(r_\theta)_{\theta \in S^1}$.

The induced action on $H_1(M,\mathbb{R}) \simeq \mathbb{R}^{2g}$ by the Teichmüller flow $(a_t)_{t\in\mathbb{R}}$ on $\mathcal{M}_g$ defines so called **Kontsevich-Zorich cocycle** $A^{KZ} : \mathbb{R} \times \mathcal{M}_g \to SL_{2g}(\mathbb{R})$.

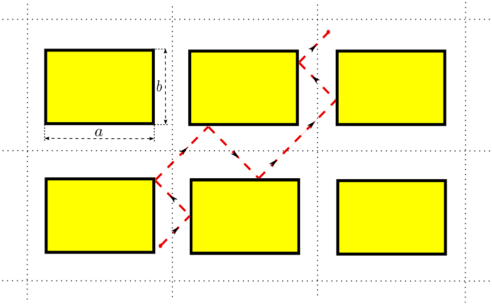
Theorem (Chaika-Eskin, 2015)

For every translation surface (M,ω) for almost every $\theta \in S^1$ the rotated surface $(M, r_\theta\omega)$ is Birkhoff (for the Teichmüller flow) and Oseledets (for the Kontsevich-Zorich cocycle) generic.

This result is essentially based on the recent celebrated result of Eskin and Mirzakhani providing a classification of $SL_2(\mathbb{R})$-orbit closures in the moduli space of translation surfaces.
The Chaika-Eskin result is crucial for full understanding of the dynamical properties of the periodic version of the Ehrenfest wind-tree model.

Ehrenfest wind-tree model

The Ehrenfest wind-tree model is the billiard flow on $\mathbb{R}^2$ with $\mathbb{Z}^2$-periodic array of rectangular scatterers, whose sides are vertical and horizontal and their lengths are given by parameters $(a, b) \in (0, 1)^2$. The billiard table is denoted by $E(a, b)$.



Theorem (F-Ulcigrai, 2014)

For every pair $(a,b)\in(0,1)^2$ and a.e. direction $\theta\in S^1$ the directional billiard flow $(b_t^\theta)_{t\in\mathbb{R}}$ on $E(a,b)$ is not ergodic.

Theorem (Delecroix-Hubert-Lelievre, 2014)

For every pair (a,b) and for a.e. direction $\theta\in S^1$ the speed of diffusion of $(b_t^\theta)_{t\in\mathbb{R}}$ on $E(a,b)$ is of order $t^{2/3}$, i.e. for a.e. $x\in E(a,b)$ we have

$$\limsup_{|t|\to+\infty}\frac{\log d(x,b_t^\theta x)}{\log t}=2/3.$$

Both results were originally proved for almost every choice of parameters (a,b). Thanks to the Chaika-Eskin theorem they can be fully proved.

Homogeneous dynamics

We will deal with dynamical systems simpler than Teichmüller. More precisely, flows on homogenous spaces. Denote by $ASL_2(\mathbb{R}) := SL_2(\mathbb{R}) \ltimes \mathbb{R}^2$ the affine special linear group which is a semidirect product with the multiplication given by

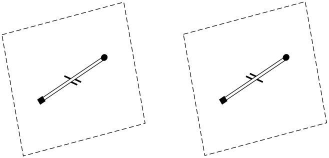
$$(h_1, \xi_1) \cdot (h_2, \xi_2) = (h_1 h_2, h_1 \xi_2 + \xi_1).$$

For any lattice $\Gamma \subset ASL_2(\mathbb{Z}) := SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ in $ASL_2(\mathbb{R})$ let us consider the quotient space $X_\Gamma = ASL_2(\mathbb{R})/\Gamma$ equipped with left invariant Haar measure μ_Γ. Next denote by $(a_t)_{t\in\mathbb{R}}$ so called geodesic flow on X_Γ defined by the left multiplication by

$$\left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right), \quad t \in \mathbb{R}.$$

This flow is ergodic on (X_Γ, μ_Γ).

- If $\Gamma = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ then $X_\Gamma = ASL_2(\mathbb{R})/\Gamma$ is the space of affine lattices in $\mathbb{R}^2$.
- If $\Gamma = SL_2(\mathbb{Z}) \ltimes 2\mathbb{Z}^2$ then $X_\Gamma = ASL_2(\mathbb{R})/\Gamma$ is the moduli space of two point branched covers of translation tori.



Then the geodesic flow $(a_t)_{t\in\mathbb{R}}$ coincides with the Teichmüller flows and we can transfer the notion of the Kontsevich-Zorich cocycle to X_Γ. Then $A^{KZ} : \mathbb{R} \times X_\Gamma \to SL_4(\mathbb{R})$.

The main result of the talk yields a more general version of the Chaika-Eskin result but in less general context of the homogeneous space X_Γ. Then some applications will be presented.

Theorem

Suppose that $\phi : [0,1] \to ASL_2(\mathbb{R})$ *is a* C^2*-curve of the form*

$$\phi(s) = \left(\begin{pmatrix} h_{11}(s) & h_{12}(s) \\ h_{21}(s) & h_{22}(s) \end{pmatrix}, \begin{pmatrix} v_1(s) \\ v_2(s) \end{pmatrix} \right),$$

so that the determinant of the Wronskian matrix

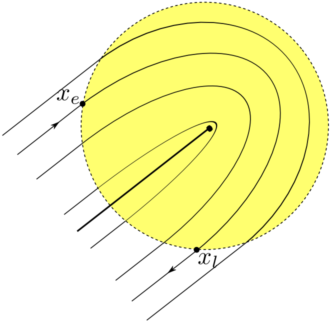
$$\begin{pmatrix} h_{11}(s) & h_{12}(s) & v_1(s) \\ h'_{11}(s) & h'_{12}(s) & v'_1(s) \\ h''_{11}(s) & h''_{12}(s) & v''_1(s) \end{pmatrix}$$

is non-zero for a.e. $s \in [0,1]$*. Then given* $(h,\xi) \in ASL_2(\mathbb{R})$ *and any lattice* $\Gamma \subset ASL_2(\mathbb{Z})$ *almost every element of the curve*

$$[0,1] \ni s \mapsto \phi(s)(h,\xi)\Gamma \in X_\Gamma \text{ is Birkhoff generic.}$$

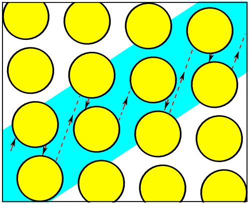
Moreover, if $\Gamma = SL_2(\mathbb{Z}) \ltimes 2\mathbb{Z}^2$ *then* $\phi(s)(h,\xi)\Gamma$ *is also Oseledets generic (for the Kontsevich-Zorich cocycle) for a.e.* $s \in [0,1]$*.*

Let us consider a round lens of radius $R > 0$ with nonconstant refractive index (RI). For the Eaton lens the RI varies depending on the distance from the center of the lens from 1 to infinity and it is given by $RI = \sqrt{\frac{2R}{r} - 1}$ in polar coordinates. The RI is 1 outside of the lens. It has a singularity at the center of the lens that the RI goes to infinity. For such type of lens the direction of the light motion is reversed after passing through the lens.



Lattices of Eaton lenses

Let us consider identical Eaton lenses arranged on the plane on a lattice Λ. Our aim is to study the light orbits for such periodic system of lens, denoted by $L(\Lambda, R)$. Note that for every light ray there is a direction $\theta \in [0, \pi]$ such that the light ray flows in direction θ or $-\theta$ outside the lenses then θ is called the direction of the light ray. F-Schmoll (2014) observed the phenomenon that light rays in $L(\Lambda, R)$ are often trapped inside bands of finite width.



We say that a direction θ is **trapped** if there exists $C(\theta) > 0$ and $v(\theta) \in S^1$ such that every light ray on $L(\Lambda, R)$ in direction θ is trapped in an infinite band of width $C(\theta)$ and in the direction $v(\theta)$.

Lattices of Eaton lenses

Theorem (F-Schmoll, 2014)

For every $0 < R < \sqrt{2\sqrt{3}}$ for almost every choice of unimodular lattice $\Lambda \subset \mathbb{R}^2$ almost every direction $\theta \in S^1$ is trapped in the system $L(\Lambda, R)$.

Theorem (F-Shi-Ulcigrai, 2015)

For every $0 < R < \sqrt{2\sqrt{3}}$ for every unimodular lattice $\Lambda \subset \mathbb{R}^2$ almost every direction $\theta \in S^1$ is trapped in the system $L(\Lambda, R)$.

The proof relies on Birkhoff and Oseledets genericity in $ASL_2(\mathbb{R})/SL_2(\mathbb{Z}) \ltimes 2\mathbb{Z}^2$ (the moduli space of two point branched covers of tori) for the curves

$$\phi(s) = \left(\begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}, \begin{pmatrix} 2R \\ 0 \end{pmatrix} \right)$$

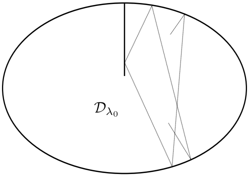
Remark: Chaika-Eskin result is useless.

Billiards in ellipses with barriers

Let $0 < b < a$ and denote by $\{\mathcal{C}_\lambda : 0 < \lambda < a\}$ the family of confocal ellipses (for $\lambda \leq b$) and hyperbolas (for $b < \lambda < a$)

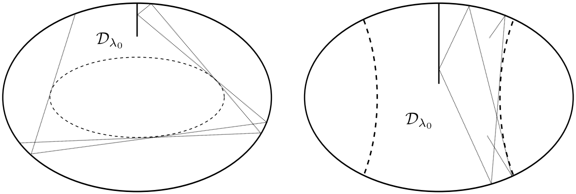
$$\frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} = 1.$$

Let us consider the billiard flow inside the ellipse $\mathcal{C}_0$ with one linear vertical obstacle of length $0 < \lambda_0 < \sqrt{b}$, which is positioned as shown in figure.



Such type of billiards was recently introduced by Dragović and Radnović (2014) as a new class of pseudo-integrable billiards.

The phase space of the billiard flow on $\mathcal{D}_{\lambda_0}$ splits into invariant subsets $\{\mathcal{S}_\lambda, 0 < \lambda < a\}$ so that the ellipse $\mathcal{C}_\lambda$ for $0 < \lambda \leqslant b$ or the hyperbola $\mathcal{C}_\lambda$ for $b < \lambda < a$ is a caustic of all billiard trajectories in $\mathcal{S}_\lambda$ (trajectories are tangent to $\mathcal{C}_\lambda$).



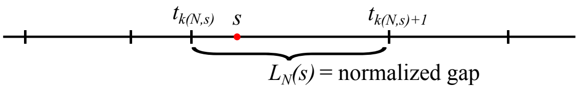
We answer affirmatively the natural conjecture (raised e.g. by Zorich) that for almost every parameters $\lambda \in (0, a)$ the billiard flow on $\mathcal{D}_{\lambda_0}$ restricted to the invariant region $\mathcal{S}_\lambda$ is uniquely ergodic. This yield density and uniform distribution of all orbits in $\mathcal{S}_\lambda$. The proof relies on Birkhoff genericity in $ASL_2(\mathbb{R})/SL_2(\mathbb{Z}) \ltimes 2\mathbb{Z}^2$. The function ϕ is given by elliptic integrals.

Let us considered the sequence $\{\sqrt{n} \mod 1\}_{n\geqslant 1}$ of *fractional parts of square roots* which is uniformly distributed in $[0,1]$. The next step of understanding the randomness of such sequence is to study its gap distribution. It was done by Elkies and McMullen (2004). For any natural $N \geq 1$ and let $0 = t_1 \leq t_2 \leq \ldots, \leq t_N < 1$ be increasingly ordered fractional parts of $\{\sqrt{1}, \sqrt{2}, \ldots, \sqrt{N}\}$. The gap distribution describes the limit behaviors as $N \to \infty$ of renormalized consecutive gaps $\{(t_{n+1} - t_n)/(1/N)\}_{1\leqslant n\leq N}$. Elkies-McMullen proved that there is a continuous probability density $F : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ (real analytic over three intervals with explicit formulas) such that for every interval $[a, b] \subset \mathbb{R}_{\geq 0}$

$$\frac{1}{N}\#\{1 \leqslant n \leqslant N : (t_{n+1} - t_n)N \in [a, b]\} \to \int_a^b F(s)\, ds \text{ as } N \to \infty.$$

We deal with related problem asked by J. Marklof: to understand the distribution of normalized gaps containing a fixed $s \in [0,1]$. Denote by $k(N,s)$ the largest $1 \leqslant k \leqslant N$ with $t_k \leqslant s$. Then we define

$$L_N(s) = N(t_{k(N,s)+1} - t_{k(N,s)}).$$



For every $s \in [0,1]$ which is not the fractional part of the square root of a natural number, $L_N(s)$ is the normalized length of the gap of the fractional parts of $\{\sqrt{1}, \ldots, \sqrt{N}\}$ which contains s.
For fixed $s \in [0,1]$ the sequence $(L_N(s))_{N\geqslant 1}$ is very slowly varying (is constant on long intervals in time) we deal with its subsequences, more precisely we study $(L_{r_n}(s))_{n\geqslant 1}$, where $(r_n)_{n\geqslant 1}$ is a geometric progression.

Theorem

If $(r_n)_{n\geqslant 1}$ is a geometric progression then for Lebesgue almost every $s \in [0,1]$ the sequence $(L_{r_n}(s))_{n\geqslant 1}$ converges in distribution to the probability measure $tF(t)\,dt$ on $\mathbb{R}_{\geq 0}$, i.e. for every interval $[a,b] \subset \mathbb{R}_{\geq 0}$

$$\frac{1}{N}\#\{1 \leqslant n \leqslant N : L_{r_n}(s) \in [a,b]\} \to \int_a^b tF(t)\,dt \quad \text{as } N \to \infty.$$

The proof relies on two steps. First we use a connection of such kind of problems with homogeneous dynamics on $ASL_2(\mathbb{R})/ASL_2(\mathbb{Z})$ (the space of affine lattices) established by Elkies and McMullen. Then we apply Birkhoff genericity of almost every point of the curve in $ASL_2(\mathbb{R})/ASL_2(\mathbb{Z})$ given by

$$\phi(s) = \left(\begin{pmatrix} 1 & -2s \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} s^2 \\ s \end{pmatrix}\right).$$