Rates of decay associated with operator semigroups

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A damped wave equation

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} - \Delta u + 2a(x)\frac{\partial u}{\partial t} &= 0 \qquad (t > 0, x \in \Omega)\\ u(x,t) &= 0 \qquad (t > 0, x \in \partial\Omega)\end{aligned}$$

$$u(\cdot, 0) = u_0 \in H_0^1(\Omega), \qquad \frac{\partial u}{\partial t}(\cdot, 0) = u_1 \in L^2(\Omega).$$

Here, Ω is a (smooth) bounded domain in $\mathbb{R}^n$, and $a : \Omega \to [0, \infty)$ (continuous).

Energy

$$E(u,t) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \left|\frac{\partial u}{\partial t}\right|^2 \right) dx,$$

decreasing in t.

Energy decay

Except in degenerate cases,

1. the energy $E(u,t) \to 0$ as $t \to \infty$;
2. if the domain of damping $\{x : a(x) > 0\}$ satisfies the geometric optics condition then the decay occurs at an exponential rate (Bardos-Lebeau-Rauch, 1992);
3. in other cases, the decay occurs at a polynomial rate or a logarithmic rate, uniformly for smooth initial data.

Operator formulation

Reformulate the damped wave equation:

$$\begin{aligned} X &= H_0^1(\Omega) \times L^2(\Omega), \\ A &= \begin{pmatrix} 0 & 1 \\ \Delta & -2a(x) \end{pmatrix}, \\ D(A) &= (H^2 \cap H_0^1) \times H_0^1. \end{aligned}$$

$$U(t) = \begin{pmatrix} u(t) \\ \frac{\partial u}{\partial t} \end{pmatrix} \in X, \qquad E(u,t) = \tfrac{1}{2}\|U(t)\|^2_{H^1 \times L^2},$$

$$U'(t) = AU(t), \qquad U(0) = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

Lebeau (1996) established that

$$\sigma(A) \subseteq \{\lambda \in \mathbb{C} : -2\|a\|_\infty \le \operatorname{Re}\lambda < 0\},$$

and $\left\|(is - A)^{-1}\right\|$ grows (at most) exponentially as $|s| \to \infty$.

The damped wave equation is well-posed, so A generates a C_0-semigroup of contractions $\{T(t) : t \ge 0\}$ on X, a strongly continuous family of operators such that

$$\begin{aligned} (\lambda - A)^{-1} &= \int_0^\infty e^{-\lambda t} T(t)\, dt \qquad (\operatorname{Re}\lambda > 0), \\ U(t) &= T(t)(u_0, u_1). \end{aligned}$$

Decay of $E(u,t)$ uniformly for initial data $(u_0, u_1) \in D(A)$ corresponds to decay of $\|T(t)(\lambda - A)^{-1}\|$ for any $\lambda \in \rho(A)$.

Estimates for the rate of decay were obtained from estimates for the growth of $\|(is - A)^{-1}\|$:
Lebeau, Burq, Batkai-Engel-Prüss-Schnaubelt, Liu–Rao, ...

Anosov flows

Let $\mathcal{M}$ be a (compact) Riemannian manifold (without boundary), and $\{\varphi_t : t \in \mathbb{R}\}$ be a smooth flow on $\mathcal{M}$. So-called *Anosov flows* have hyperbolic behaviour on the tangent spaces of $\mathcal{M}$, chaotic behaviour on $\mathcal{M}$, and mixing properties, i.e. for smooth f, g,

$$\int_{\mathcal{M}} f.(g \circ \varphi_t) \to \int_{\mathcal{M}} f \int_{\mathcal{M}} g \qquad (t \to \infty).$$

What is the rate of convergence?

Ruelle, Pollicott, Chernoff, Dolgopyat, Liverani, Tsujii, Butterley, Faure, Sjöstrand, Dyatlov, Zworski,.....

Dolgopyat estimate

$$C^\infty(\mathcal{M}) \subset X \subset C(\mathcal{M})$$

$$T(t)f = f \circ \varphi_t, \qquad \text{generator } A$$

Ingredients:

1. Quasi-compactness argument for spectral gap
2. Resolvent estimate for large $|s|$

$$\left\|(\alpha + is - A)^{-\gamma \log |s|}\right\| \leq C(\lambda + \alpha)^{-\gamma \log |s|}. \qquad \text{(Dol)}$$

(fixed $\alpha, \gamma, \lambda, C > 0$ with $\gamma(\lambda + \alpha) < 1$)

(Dol) implies

$$\|(is - A)^{-1}\| = O(\log |s|)$$

and a polynomial bound for $-\lambda/2 < \operatorname{Re} z < 0$.

Ingham-Karamata theorem

X complex Banach space, $f : \mathbb{R}_+ = [0, \infty) \to X$

Laplace transform:

$$\widehat{f}(\lambda) = \int_0^\infty f(t) e^{-\lambda t}\, dt \qquad (\operatorname{Re} \lambda > 0)$$

Theorem (Ingham, Karamata, 1933-35)

Let $f \in L^\infty(\mathbb{R}_+, X)$, and assume that $\widehat{f}$ extends analytically at each point of $i\mathbb{R}$. Then

$$\lim_{t\to\infty} \int_0^t f(s)\, ds = \widehat{f}(0).$$

Semigroup version

From now on,
$\{T(t) : t \geq 0\} \subset \mathcal{B}(X)$, bounded C_0-semigroup on X
Generator A, unbounded, closed; $\sigma(A) \subseteq \{\operatorname{Re}\lambda \leq 0\}$.

$$T(t)T(s) = T(t+s),$$
$$(\lambda I - A)^{-1}x = (T(\cdot)x)\widehat{\ }(\lambda) = \int_0^\infty e^{-\lambda t} T(t)x\,dt,$$
$$\|T(t)\| \leq K$$

Theorem

Assume that $\{T(t) : t \geq 0\}$ is bounded, and that $\sigma(A) \cap i\mathbb{R}$ is empty. Then

$$\lim_{t\to\infty} \|T(t)A^{-1}\| = 0.$$

Hence

$$\lim_{t\to\infty} \|T(t)x\| = 0 \qquad (x \in X).$$

Ingham-Karamata with rates

$M : \mathbb{R}_+ \to (0,\infty)$, continuous, increasing.

$M_{\log}(s) = M(s)(\log(1+M(s)) + \log(1+s))$.

Theorem ($M_{\log}$-theorem; B-Duyckaerts, 2008)

Let $f \in L^\infty(\mathbb{R}_+, X)$, and assume that $\widehat{f}$ extends analytically to

$$\Omega_M := \left\{ \lambda \in \mathbb{C} : \operatorname{Re}\lambda > -\frac{1}{M(|\operatorname{Im}\lambda|)} \right\},$$

and

$$\|\widehat{f}(\lambda)\| \le M(|\operatorname{Im}\lambda|), \qquad (\lambda \in \Omega_M).$$

Then, for certain $c > 0$,

$$\left\| \widehat{f}(0) - \int_0^t f(s)\,ds \right\| \le \frac{C}{M_{\log}^{-1}(ct)}.$$

Corollary

Suppose that

$$\|(is - A)^{-1}\| \le M(|s|) \quad (s \in \mathbb{R}).$$

Then, for certain $c > 0$*,*

$$\|T(t)A^{-1}\| = O\left(\frac{1}{M_{\log}^{-1}(ct)}\right) \quad (t \to \infty),$$

where

$$M_{\log}(s) = M(s)\left(\log(1 + M(s)) + \log(1 + s)\right).$$

Exponential decay

Assume that A generates a bounded C_0-semigroup $\{T(t) : t \geq 0\}$ on a Banach space, and that $\|(\lambda - A)^{-1}\|$ is bounded for $\operatorname{Re}\lambda = 0$. Then it is also bounded for $\operatorname{Re}\lambda \geq -\delta$ for some $\delta > 0$.

The $M_{\log}$-theorem implies that $\|T(t)A^{-1}\| \leq Ce^{-ct}$.

Proved by Weis and Wrobel in 1996, without assuming the semigroup is bounded.

In Hilbert spaces, the Gearhart-Prüss Theorem gives $\|T(t)\| \leq Ce^{-ct}$.

Logarithmic decay

Now T is any bounded semigroup and $\sigma(A) \cap i\mathbb{R}$ is empty.

Suppose that

$$\left\|(is - A)^{-1}\right\| \le C \exp(C|s|) \quad (s \in \mathbb{R}).$$

Then

$$\left\|T(t)A^{-1}\right\| = O\left(\frac{1}{\log t}\right) \quad (t \to \infty).$$

Proved by Burq (1998) for Hilbert spaces

Polynomial decay

Assume that

$$\left\|(is - A)^{-1}\right\| \leq C(1 + |s|)^{\alpha} \quad (s \in \mathbb{R}).$$

Then

$$\left\|T(t)A^{-1}\right\| = O\left(\left(\frac{\log t}{t}\right)^{1/\alpha}\right).$$

Improves results of Batkai et al (Banach spaces) and Liu–Rao (Hilbert space)

Enhanced version

Theorem (B-Borichev-Tomilov)

Let $f \in L^p(\mathbb{R}_+, X)$, where $1 \le p \le \infty$, and assume that $\widehat{f}$ extends analytically to Ω_M and, for some $\alpha, \beta > 0$,

$$\|\widehat{f}(\lambda)\| \le C(1 + |\operatorname{Im}\lambda|)^\alpha M(|\operatorname{Im}\lambda|)^\beta, \qquad \lambda \in \Omega_M,$$

Then there exists $c > 0$, depending on p, α, β, such that the function

$$t \mapsto M_{\log}^{-1}(ct)\left(\widehat{f}(0) - \int_0^t f(s)\,ds\right)$$

belongs to $L^p(\mathbb{R}_+, X)$.

The shape of Ω_M is much more important than the bound on $\widehat{f}$.

Back to Anosov and Dolgopyat

Apply the previous theorem, with $p = \infty$, $M(s) = 1$, $f(t) = T(t)x$ for a bounded semigroup.

Assume

- $\sigma(A) \subset \{\operatorname{Re} \lambda \le -\omega\}$ for $\omega > 0$ (spectral gap)
- $\|(a + is - A)^{-1}\| \le C(1 + |s|)^\alpha \quad (-\omega < a < 0)$,

Then $\|T(t)A^{-1}\| \le C' e^{-ct}$.

Hence Dolgopyat estimates imply exponential decay.

Necessary condition

Theorem

Let $m : (0, \infty) \to (0, \infty)$ *be decreasing with* $\lim_{t\to\infty} m(t) = 0$. *Assume that*

$$\left\| T(t)(1-A)^{-1} \right\| \le m(t) \quad (t > 0).$$

Then $\sigma(A) \cap i\mathbb{R}$ *is empty, and, for each* $c \in (0,1)$,

$$\left\| (is - A)^{-1} \right\| = O\left(m^{-1}(c/|s|) \right) \quad (|s| \to \infty).$$

So the apparently optimal rate of decay in Ingham-Karamata Theorem would have M^{-1} instead of $M_{\log}^{-1}$. Can one achieve this?

Corollary

Assume that the damped wave equation satisfies

$$E(u,t) \le m(t)^2 E(u,0)$$

for all classical solutions u, and some decreasing function $m(t)$. Then

$$\int_0^\infty \left| \frac{d}{dt} E(u,t) \right| M_{\log}^{-1}(kt)^2 \, dt < \infty$$

where $M(s) = m^{-1}(c/s)$, for some $c, k > 0$.

Theorem (Borichev-Tomilov 2010)

Let $M(s) = C(1+s)^\alpha$. In the cases of scalar functions and semigroups on Banach spaces, it is not possible to improve the conclusion of the $M_{\log}$-theorem that the rate of decay is

$$O\left(\left(\frac{\log t}{t}\right)^{\frac{1}{\alpha}}\right) \quad (t \to \infty).$$

In the case of a C_0-semigroup on a Hilbert space satisfying

$$\left\|(is - A)^{-1}\right\| \le C(1+|s|)^\alpha \quad (s \in \mathbb{R}).$$

one has

$$\left\|T(t)A^{-1}\right\| = O\left(\frac{1}{t^{\frac{1}{\alpha}}}\right) \quad (t \to \infty).$$

Subexponential decay

When $M(s) = O(\log s)$, the $M_{\log}$-theorem gives as rate of decay

$$O\left(e^{-c\sqrt{t}}\right)$$

When can this be improved to the optimal rate $O\left(e^{-ct}\right)$?

Regularly varying case

M is *regularly varying* if $M(s) \sim \dfrac{s^\alpha}{\ell(s)}$ where ℓ is slowly varying.

This means that for all $\lambda > 0$,

$$\lim_{s\to\infty} \frac{M(\lambda s)}{M(s)} = \lambda^\alpha, \qquad \lim_{s\to\infty} \frac{\ell(\lambda s)}{\ell(s)} = 1.$$

For example,

- $\ell(s) = (\log s)^\beta \quad (\beta \in \mathbb{R})$
- $\ell(s) = \exp\left((\log s)^\beta\right) \quad (0 < \beta < 1)$
- $\ell(s) = \exp\left(\dfrac{\log s}{\log\log s}\right)$

Consider the case $\alpha = 1$ (purely for simplicity), ℓ increasing.

Theorem (B-Chill-Tomilov, to appear, JEMS)

Assume that X is a Hilbert space, and ℓ is slowly varying and increasing, and

$$\|(is - A)^{-1}\| = O\left(\frac{|s|}{\ell(|s|)}\right) \quad (|s| \to \infty).$$

Then

$$\|T(t)A^{-1}\| = O\left(\frac{1}{t\ell(t)}\right) \quad (t \to \infty).$$

For many (but not all) ℓ, this gives the optimal result $\|T(t)A^{-1}\| = O(1/M^{-1}(t))$.

Theorem (B-Chill-Tomilov, to appear, JEMS)

Assume that X is a Hilbert space, and ℓ is slowly varying and decreasing, and

$$\left\|(is - A)^{-1}\right\| = O\left(\frac{|s|}{\ell(|s|)}\right) \quad (|s| \to \infty).$$

Then, for every $\varepsilon > 0$,

$$\left\|T(t)A^{-1}\right\| = O\left(\frac{(\log t)^{\varepsilon}}{t\widetilde{\ell}(t)}\right) \quad (t \to \infty).$$

Here $\widetilde{\ell}$ is a slowly varying function which is sometimes, but not always, the same as ℓ. However the optimal result would have $\varepsilon = 0$.

Outline proof of Ingham-Karamata

$$\widehat{f}(0) - \int_0^t f(s)\,ds$$
$$= \frac{1}{2\pi i}\int_\gamma \left(1+\frac{z^2}{R^2}\right)\left(\widehat{f}(z) - \int_0^t e^{-zs} f(s)\,ds\right) e^{tz}\,\frac{dz}{z}.$$

Main estimate

$$\left\| \widehat{f}(0) - \int_0^t f(s)\,ds \right\| \le \frac{2\|f\|_\infty}{R} + \frac{1}{2\pi}\left\| \int_{\gamma'} \left(1+\frac{z^2}{R^2}\right)\widehat{f}(z) e^{tz}\,\frac{dz}{z} \right\|.$$

Choose γ' and R carefully, as functions of t.

Outline of proof: polynomial case

Semigroup on Hilbert space, $M(s) = C(1+s)^\alpha$.

- Use complex analysis to show

$$\|(\lambda - A)^{-1}(-A)^{-\alpha}\| \le C \quad (\operatorname{Re}\lambda \ge 0).$$

- Use Plancherel's Theorem to show

$$\|T(t)(-A)^{-\alpha}\| \le Ct^{-1}.$$

- Use semigroup property and interpolation (moment inequality) to show

$$\|T(t)(-A)^{-1}\| \le Ct^{-1/\alpha}.$$

Outline of proof: regularly varying case

Hilbert space, $M(s) \sim \frac{s^\alpha}{\ell(|s|)}$, $\alpha > 0$, ℓ increasing.

- Find a complete Bernstein function f_ℓ, as large as possible, such that

$$\|(\lambda - A)^{-1} A^{-(\alpha-1)} f_\ell(-A^{-1})\| \leq C \quad (\operatorname{Re}\lambda \geq 0).$$

  For example, $f_{\log}(-A^{-1}) = -A^{-1}\log(I - A)$.

- Use Plancherel's Theorem to show

$$\|T(t) A^{-(\alpha-1)} f_\ell(-A^{-1})\| \leq C t^{-1}.$$

- Use an interpolation inequality for complete Bernstein functions of semigroup generators, together with an Abelian/Tauberian theorem for Stieltjes transforms (Karamata, 1930s), to remove the log term in the $M_{\log}$ result.

Complete Bernstein functions

$f : (0, \infty) \to (0, \infty)$ is a *complete Bernstein function* if

$$f(s) = a + bs + \int_{(0,\infty)} \frac{s}{s+u} \, d\nu(u) \qquad (s > 0)$$

for some $a, b \geq 0$ and positive measure ν on $(0, \infty)$.

Equivalently, f extends analytically to $\mathbb{C} \setminus (-\infty, 0]$ mapping the upper half-plane to itself, and $\lim_{s \to 0+} f(s)$ exists and is real [i.e, f is a Nevanlinna-Pick function and is positive on the positive axis.]

Equivalently, $f(s) = S(1/s)$, where

$$S(s) = a + \frac{b}{s} + \int_{(0,\infty)} \frac{1}{s+u} \, d\nu(u).$$

This integral is the *Stieltjes transform* of the measure ν (or of its distribution function).

Interpolation inequality

Theorem

Let A be the generator of a bounded C_0-semigroup $(T(t))_{t \geq 0}$ on a Banach space, and assume that A is invertible. There exists a constant $c > 0$ such that the following holds for all complete Bernstein functions f and all $t \geq 0$:

$$\|T(t)f(-A^{-1})\| \geq c \frac{\|T(2t)A^{-1}\|}{\|T(t)A^{-1}\|} f(\|T(t)A^{-1}\|).$$

Example of L^2-version

X Hilbert space, $T(t)$ contractions

Assume that $D(A) = D(A^*)$. Consider $-(A + A^*)$, symmetric, non-negative.

Let S be any non-negative, self-adjoint extension of $-(A + A^*)$, $B = S^{1/2}$

Theorem

Assume in addition that $\sigma(A) \cap i\mathbb{R}$ is empty, and $\|R(is, A)\| \le M(|s|)$. Then, for all $x \in X$,

$$\int_0^\infty M_{\log}^{-1}(kt)^2 \|BT(t)A^{-1}x\|^2 \, dt < \infty.$$