Convergence of Nonconventional Ergodic Averages

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Double Recurrence Wiener-Wintner averages

Recall that given a measure-preserving system $(X, \mathcal{F}, \mu, T)$, the Cesaro averages of $f_1(T^{an}x)f_2(T^{bn}x)$ converge for μ-a.e. $x \in X$ due to Bourgain [8].

Question: Can this result be extended to Wiener-Wintner type averages?

In 2001, D. Duncan provided a Wiener-Wintner extension of Bourgain's result in his Ph.D. Thesis [11] before the appearance of Host-Kra-Ziegler factors.

Duncan's Ph.D. Thesis

Theorem 1 (Duncan [11], 2001)

Let $(X, \mathcal{F}, \mu, T)$ be a standard ergodic system, T totally ergodic transformation, $f_1, f_2 \in L^2(X)$, and $\mathcal{CL}$ be the maximal isometric extension of the Kronecker factor of T. Let

$$W_N(f_1, f_2, x, t) = \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{an}x) f_2(T^{bn}x) e^{2\pi i n t}.$$

1. *If either f_1 or f_2 belongs to $\mathcal{CL}^\perp$, then there exists a set of full measure X_{f_1,f_2} such that for all $x \in X_{f_1 \otimes f_2}$,*

$$\limsup_{N\to\infty} \sup_{t\in\mathbb{R}} |W_N(f_1, f_2, x, t)| = 0$$

2. *If $f_1, f_2 \in \mathcal{CL}$, then for μ-a.e. $x \in X$, $W_N(f_1, f_2, x, t)$ converges for all $t \in \mathbb{R}$, provided that the cocycle associated with $\mathcal{CL}$ is affine.*

Updates in 2014

Remark: $\mathcal{CL}$ used to be called the "Conze-Lesigne" factor.

In 2014,David Duncan, Ryo Moore (my current Ph.D. student), and I generalized in the following way:

- The transformation T is just ergodic instead of totally ergodic.
- The characteristic factor is now the second Host-Kra-Ziegler factor (or the "new" Conze-Lesigne factor), $\mathcal{Z}_2$.
- The assumption about the cocycle is dropped.

Double Recurrence Wiener-Wintner averages

Theorem 2 (A., Duncan, Moore 2014, [4], to appear on *ETDS*)

Let $(X, \mathcal{F}, \mu, T)$ be a standard ergodic dynamical system. Let $f_1, f_2 \in L^\infty(X)$. Let $W_N(f_1, f_2, x, t) = \frac{1}{N}\sum_{n=0}^{N-1} f_1(T^{an}x)f_2(T^{bn}x)e(nt)$. Then there exists a set of full-measure X_{f_1,f_2} such that for all $x \in X_{f_1,f_2}$ and for any $t \in \mathbb{R}$, the sequence $W_N(f_1, f_2, x, t)$ converges.
Also, if either f_1 or f_2 belongs to $\mathcal{Z}_2^\perp$, then for any $x \in X_{f_1,f_2}$,

$$\limsup_{N\to\infty} \sup_{t\in\mathbb{R}} |W_N(f_1, f_2, x, t)| = 0,$$

where $\mathcal{Z}_k$ is the k-th Host-Kra-Ziegler factor.

Remarks on the D.R.W.W. result

- This Wiener-Wintner result already shows that the sequence $c_n = f_1(T^{an}x)f_2(T^{bn}x)$ is μ-a.e. a good universal weight for the mean ergodic theorem, i.e. for any other dynamical system $(Y, \mathcal{G}, \nu, S)$ and $g \in L^\infty(\nu)$, the averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^{an}x)f_2(T^{bn}x)g \circ S^n$$

  converge in $L^2(\nu)$.
- The l-th Host-Kra-Ziegler factor $\mathcal{Z}_l$ of X is the inverse limit of l-step nilsystems of X.

Definition

Let G be a l-step nilpotent Lie group, and Γ be a discrete co-compact subgroup of G. Then G/Γ is called an l-step nilmanifold. A measure-preserving system $(X, \mathcal{F}, \mu, T)$, where $X = G/\Gamma$, $\mathcal{F}$ a Borel sigma-algebra with Haar measure μ, and $Tx = g \cdot x$ for some $g \in G$, is called an l-step nilsystem.

Double recurrence Wiener-Wintner results: Approach

We broke the theorem into two cases: When either f_1 or f_2 belongs to $\mathcal{Z}_2^{\perp}$, or when both of them are in $\mathcal{Z}_2$.

For the first case, we showed that the L^2-norm of the lim sup of the averages is bounded above by the (constant multiple of the) minimum of $|||f_1|||_3$ or $|||f_2|||_3$, where $|||\cdot|||_{k+1}$ is the k-th Gowers-Host-Kra seminorm [15, 16] that characterizes $\mathcal{Z}_k$.We also used the integral kernel that was seen in the work of Furstenberg and Weiss [14, Theorem 2.1, in proof]

Lemma

Let T be an ergodic map, and s be a positive integer. Then there exist a disjoint partition of T^s-invariant sets $A_1, \dots, A_l$ such that every T^s-invariant function f can be expressed as an integral with respect to the kernel $K(x,y) = l\sum_{k=1}^{l} \mathbf{1}_{A_i}(x)\mathbf{1}_{A_i}(y)$.

Polynomial action on nilmanifolds

For the second case, we used A. Leibman's pointwise convergence result on polynomial actions on a nilsystem [18].

Theorem 3 (Leibman, 2005)

Let $X = G/\Gamma$ be a nilmanifold, and $\{g(n)\}_{n\in\mathbb{Z}}$ be a polynomial sequence in G. Then for any $x \in X$ and continuous functions F on X, the average $\frac{1}{N}\sum_{n=1}^{N} F(g(n)x)$ converges.

Polynomial Wiener-Wintner and the double recurrence

We extended the Wiener-Wintner result to the following:

Theorem 4 (Assani, Moore, 2014 [5], to appear in J. Anal Math)

Let $(X, \mathcal{F}, \mu, T)$ *be a probability measure-preserving system. Let* $f_1, f_2 \in L^\infty(X)$. *Let* $W_N(f_1, f_2, x, p) = \frac{1}{N}\sum_{n=0}^{N-1} f_1(T^{an}x)f_2(T^{bn}x)e(p(n))$. *Then there exists a set of full-measure* X_{f_1,f_2} *such that for all* $x \in X_{f_1,f_2}$ *and for any* $p \in \mathbb{R}[n]$, *the sequence* $W_N(f_1, f_2, x, p)$ *converges.*
Also, if either f_1 *or* f_2 *belongs to* $\mathcal{Z}_k^\perp$, *then for* $x \in X_{f_1,f_2}$ *and a k-th degree polynomial* $p \in \mathbb{R}[n]$,

$$\limsup_{N\to\infty} \sup_{p\in\mathbb{R}[n]} |W_N(f_1, f_2, x, p)| = 0,$$

Poly. WW and the D.R. Notes

The proof for the uniformity case is done by an induction on the degree of the polynomials. The base case $k = 1$ corresponds to the Wiener-Wintner result established earlier.

The polynomial Wiener-Wintner averages (with a single function) and their uniformity were studied by E. Lesigne (1990 [19], 1993 [20]), N. Frantzikinakis (2006 [13]), and recently by T. Eisner and B. Krause (polynomial power of T, 2014 [12]). The Abramov factors were used in the work of the first two authors (in exchange of assuming that T is totally ergodic).

This provides an alternative proof to the result of Lesigne ($f_2 = \mathbf{1}_X$, $a = 1$, a character $\phi(p(n))$ instead of $e(p(n))$).

Good universal weights

Definition

We say $(X_n)_n$ is a **process** *if for all nonnegative integers $n \geq 0$, X_n is a bounded and measurable function on some probability measure space $(\Omega, \mathcal{S}, \mathbb{P})$.*

Definition

We say the sequence $(a_n)_n$ is a good universal weight for a process $(X_n)_n$ pointwise (resp. in norm) if for every probability measure-preserving space $(\Omega, \mathcal{S}, \mathbb{P})$ for which the process $(X_n)_n$ is defined, then the averages

$$\frac{1}{N} \sum_{n=0}^{N-1} a_n X_n(\omega)$$

converges for $\mathbb{P}$-a.e. $\omega \in \Omega$ (resp. in $L^2(\mathbb{P})$).

Extensions of the return times theorem

The return times theorem has been extended in multiple directions. For example...

- Rudolph (Multi-term return times theorem, 1998, [21])
- A. (Multiple recurrence and the multi-term return times theorem, 2000, [1])
- Host and Kra (Good universal weight for the norm convergence of non conventional ergodic averages, 2009, [17])

Universal Weights

More precisely, Host and Kra showed that given an ergodic dynamical system $(X, \mathcal{F}, \mu, T)$ and a function $f \in L^\infty(\mu)$, there exists a set of full-measure $X_f \subset X$ such that for any $x \in X_f$, for any positive integer k, and for any other measure-preserving system $(Y, \mathcal{G}, \nu, S)$ with $g_1, \ldots, g_k \in L^\infty(\nu)$, the averages

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) g_1 \circ S^n \cdot g_2 \circ S^{2n} \cdots g_k \circ S^{kn}$$

converge in $L^2(\nu)$.

Question: Can c_n be generalized? For instance, can we have $c_n = f_1(T^{an}x) f_2(T^{bn}x)$ for some $f_1, f_2 \in L^\infty(\mu)$?

Double recurrence and Furstenberg averages

Recently, we have shown that $a_n = f_1(T^{an}x)f_2(T^{bn}x)$ for any $a, b \in \mathbb{Z}$ distinct is a good universal weight for the Furstenberg averages.

Theorem 5 (A., Moore 2015, [6], to appear on *ETDS*)

Let $(X, \mathcal{F}, \mu, T)$ be a measure-preserving system, and suppose $f_1, f_2 \in L^\infty(\mu)$. Then there exists a set of full-measure $X_{f_1,f_2} \subset X$ such that for any $x \in X_{f_1,f_2}$, for any $a, b \in \mathbb{Z}$ distinct, for any positive integer k, and for any other dynamical system $(Y, \mathcal{G}, \nu, S)$ with functions $g_1, \ldots, g_k \in L^\infty(\nu)$, the averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^{an}x)f_2(T^{bn}x)\prod_{i=1}^{k} g_i \circ S^{in}$$

converge in $L^2(\nu)$.

Remark

- This is a strengthening of the results of Host and Kra and of Bourgain's double recurrence theorem. In fact, we can show that for any $A, B \in \mathcal{F}$, we know that there exists a set of full-measure $X_{A,B} \subset X$ such that for any $x \in X_{A,B}$, for any other measure-preserving system $(Y, \mathcal{G}, \nu, S)$ and $E \in \mathcal{G}$, the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_A(T^{an}x)\mathbf{1}_B(T^{bn}x)\nu\left(\bigcap_{i=1}^{k} S^{-in}E\right)$$

exists.

Commuting Case

Furthermore, Theorem 5 can be extended to the case with commuting transformations. This result combines and extends the previous work of Bourgain and Tao.

Theorem 6 (A., Moore 2015, [7], submitted)

Let $(X, \mathcal{F}, \mu, T)$ be a measure-preserving system, and suppose $f_1, f_2 \in L^\infty(\mu)$. Then there exists a set of full-measure $X_{f_1,f_2} \subset X$ such that for any $x \in X_{f_1,f_2}$, for any $a, b \in \mathbb{Z}$ distinct, for any positive integer k, and for any other dynamical system with commuting transformations $(Y, \mathcal{G}, \nu, S_1, \ldots, S_k)$ with functions $g_1, \ldots, g_k \in L^\infty(\nu)$, the averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^{an}x)f_2(T^{bn}x)\prod_{i=1}^{k} g_i \circ S_i^n$$

converge in $L^2(\nu)$.

Outline of the Proof of Theorem 5

The idea of the proof is to look at whether either f_1 or f_2 belongs to the orthogonal complement of $\mathcal{Z}_{k+1}(T)$, or if they are both in this factor (where k is the number of transformations from the Furstenberg averages).

For the first case, we inductively prove that there exists a set of full-measure X_1 such that the averages converge to zero by applying the spectral theorem and the Wiener-Wintner result above to show that the averages converge to 0 (on a universal set of full-measure in X).

For the second case, we look at the appropriate factors of Y.

- If one of the functions $g_1, \ldots, g_k$ belongs to $\mathcal{Z}_k(S)^\perp$, then the averages converge to 0 in norm.
- If all of them belong to $\mathcal{Z}_k(S)$, we apply Leibman's convergence theorem.

Double recurrence and nilsequence Wiener-Wintner

In 2014 Ergodic Theory Workshop at UNC-Chapel Hill, B. Weiss asked whether Theorem 2 can be extended to nilsequences.

Definition

Let $(X = G/\Gamma, \mathcal{F}, \mu, T)$ be an l-step nilsystem. If $F \in \mathcal{C}(X)$ and $g \in G$, we say the sequence $a_n = F(g^n x)$ is a basic l-step nilsequence. An l-step nilsequence is a uniform limit of basic l-step nilsequences.

We note that $e(nt) = e^{2\pi int}$ is a 1-step nilsequence, and for any real polynomial p of degree l, $e(p(n))$ is an l-step nilsequence.

Double recurrence and nilsequence Wiener-Wintner

We answered to this question positively.

Theorem 7 (A. 2015 [3], submitted)

Let $(X, \mathcal{F}, \mu, T)$ be a measure-preserving system, and $f_1, f_2 \in L^\infty(\mu)$. Then there exists a set of full-measure $X_{f_1,f_2} \subset X$ such that for any $x \in X_{f_1,f_2}$ and for any nilsequence $(b_n)_n$, the averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^{an}x)f_2(T^{bn}x)b_n$$

converge.

This result was also obtained by Zorin-Kranich independently [22].

Remarks

- The results were obtained using the techniques seen in the speaker's work on averages along cubes [2], Host and Kra's work on uniformity seminorms [17], and the integral kernel.
- The result was originally announced in R. Moore's Ph.D. oral exam, which was on 10 April 2015. The first preprint was posted on arXiv on 22 April 2015.
- P. Zorin-Kranich obtained a similar result independently, and the first preprint appeared on arXiv on 21 April 2015.

Wiener-Wintner as a background

With the recent results on the extensions of the double recurrence theorem combined, we have the following equivalence properties:

Theorem 8 (A. 2015 [3])

Let $(X, \mathcal{F}, \mu, T)$ be a measure-preserving system, $f_1, f_2 \in L^\infty(\mu)$, and a, b be distinct integers. The following statements are equivalent.

(i) *The sequence $(f_1(T^{an}x)f_2(T^{bn}x))_n$ is a good universal weight for the linear multiple recurrence averages with single transformation in norm.*

(ii) *The* **classical** *double recurrence Wiener-Wintner averages converge off a single null set.*

(iii) *The* **nilsequence** *Wiener-Wintner averages converge off a single null set.*

Next Steps

1. Nilpotent case. Can the double recurrence good universal weight results be extended to the systems $(Y, \mathcal{G}, \nu, S_1, \ldots, S_k)$, where the transformations $S_1, \ldots, S_k$ are generating a nilpotent group?

2. Positivity. Given a measure-preserving system $(X, \mathcal{F}, \mu, T)$, with some $f_1, f_2 \in L^\infty(\mu)$, does there exist a set of full-measure $X_{f_1,f_2} \subset X$ such that for any $x \in X_{f_1,f_2}$ and for any other measure-preserving system $(Y, \nu, S_1, \ldots, S_k)$ with any $E \in \mathcal{G}$ a set with positive measure, we have

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{an}x) f_2(T^{bn}x) \nu\left(\bigcap_{i=1}^{k} S_i^{-n} E\right) > 0?$$

Any properties on f_1 and f_2 (beyond the clear requirement that $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{an}x) f_2(T^{bn}x) > 0$)? What about a positive lower bound? Syndeticity?

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Thank you for the invitation!