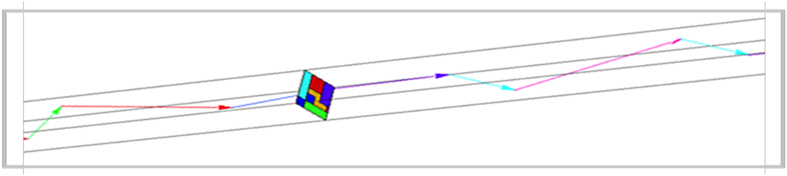
Rectilinear Polygon Exchange Transformations via Cut-and-Project Methods

Ren Yi
(joint work with Ian Alevy and Richard Kenyon)

Zero Entropy System, CIRM



Outline

1. Polygon Exchange Transformations (PETs)
2. Cut-and-Project Sets
3. (1) $+$ (2) $\Rightarrow$ Rectilinear PETs (RETs)

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3. (1) + (2) $\Rightarrow$ Rectilinear PETs (RETs)
   (Renormalization)

4. Multi-Stage RETs

5. Parameter Space of Multi-Stage RETs

What is a PET?

(Polygon Exchange Transformation)

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$$B_k = A_k + v_k, \quad \forall k = 0, \cdots, N.$$

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  For each k, the polygons A_k and B_k are translation equivalent.
- A **polygon exchange transformation (PET)** is a dynamical system on X. The map $T : X \to X$ is defined by

$$\mathbf{x} \mapsto \mathbf{x} + v_k, \quad \forall \mathbf{x} \in \mathrm{Int}(A_k).$$

What is a PET?

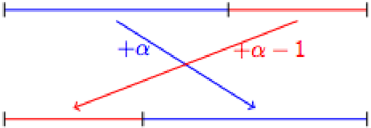

Figure: 1-dim example (interval exchange transformation)

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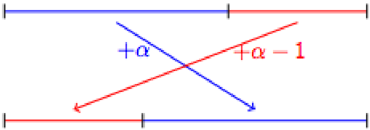

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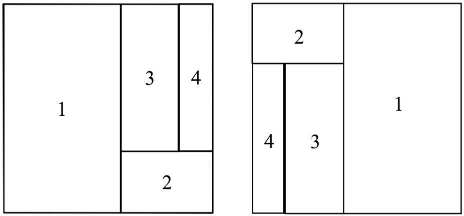

Figure: An example of PETs

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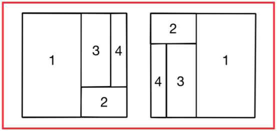

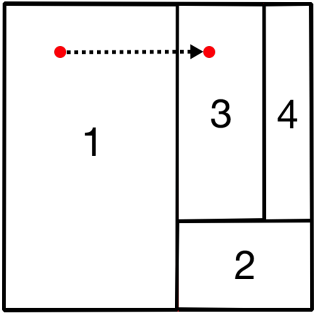

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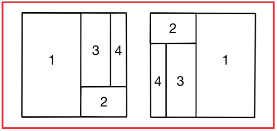

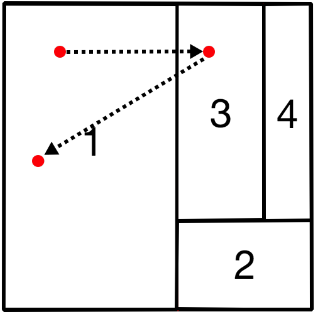

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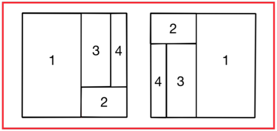

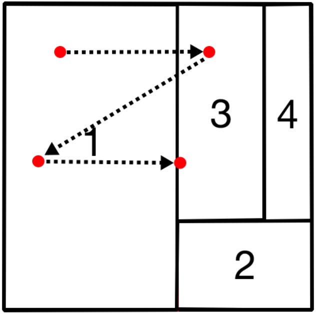

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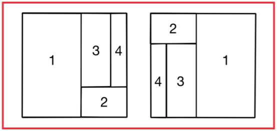

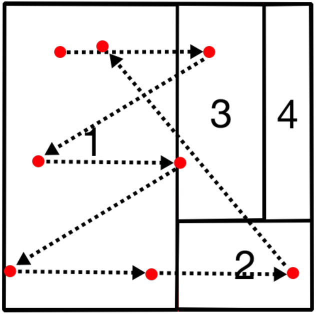

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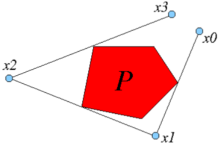

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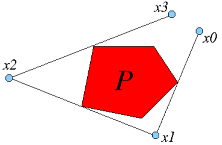

- Multigraph PETs (R. Schwartz and R. Yi)

Rectilinear PETs (RETs)

We consider the case when all $A_k \in \mathcal{A}$ and $B_k \in \mathcal{B}$ in the partitions are rectilinear polygons.

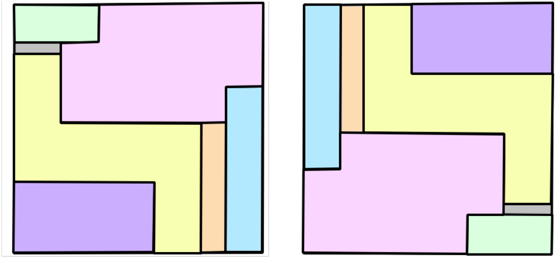

Figure: An example of rectilinear PETs

Renormalization: An approach to study PETs

Renormalization

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- Let $T : X \to X$ be a PET and Y be a subset of X. The **first return map** $\hat{T}|_Y : Y \to Y$ is given by

$$\hat{T}|_Y(p) = T^n(p), \quad \text{where } n = \min\{T^k(p) \in Y |\ k > 0\}.$$

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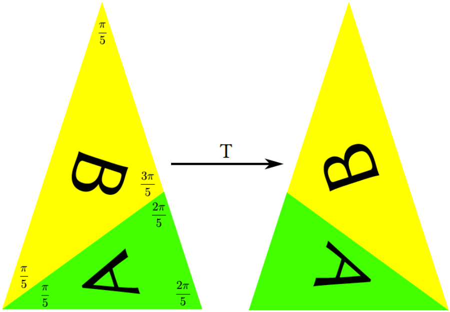
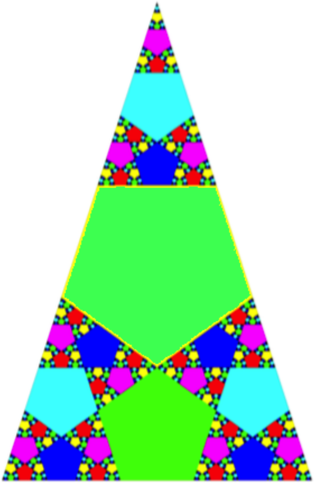
Definition (Renormalization)

A PET $T : X \to X$ is **renormalizable** if there exists a subset $Y \subset X$ such that $\hat{T}|_Y$ is conjugate to T by a affine map.

Renormalization

An example of piecewise isometric maps and its renormalization (A. Goetz).





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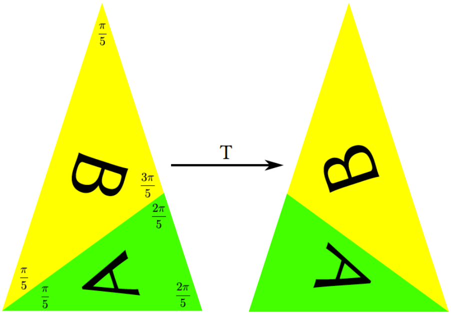

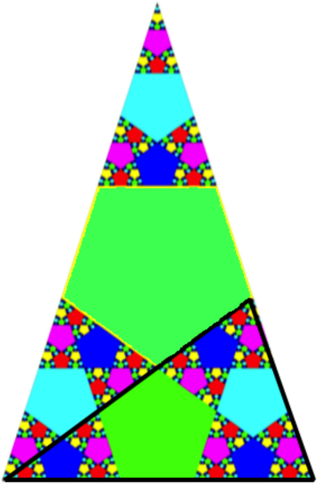

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- Eigenvectors (corresponding to λ_i):

$$\xi_i = (1, \lambda_i, \lambda_i^2).$$

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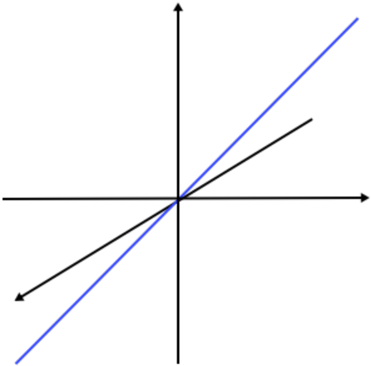

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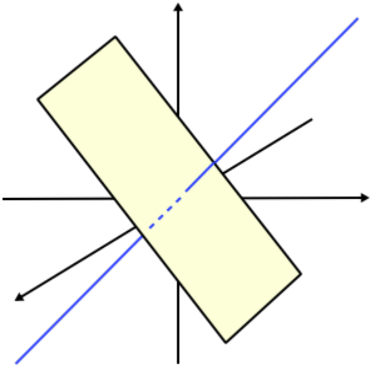

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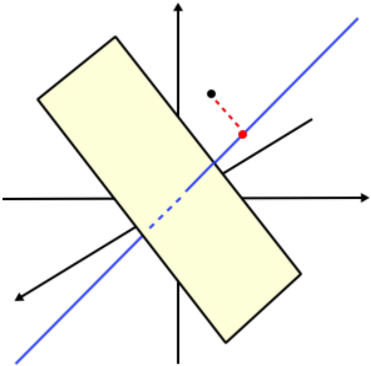

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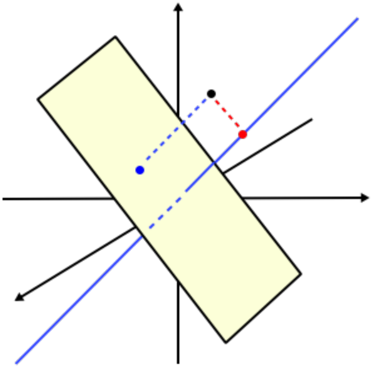

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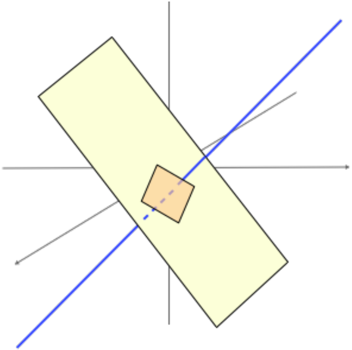

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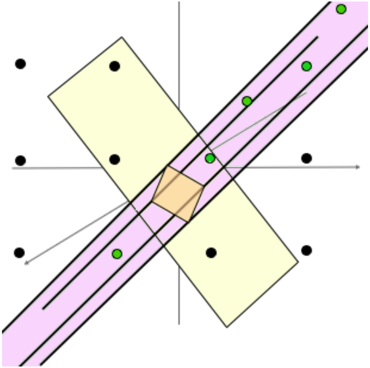

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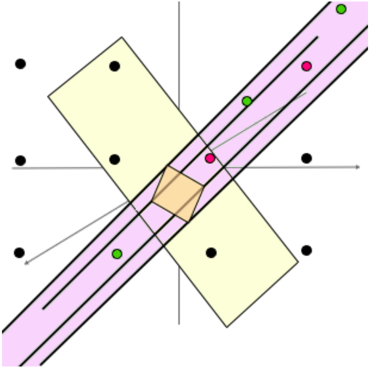

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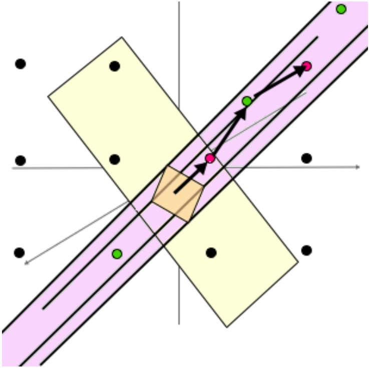

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- Let $\mathcal{E} = \{w_p - p | \ p \in \Lambda_X\}$.

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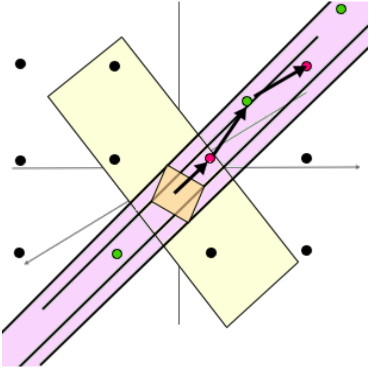

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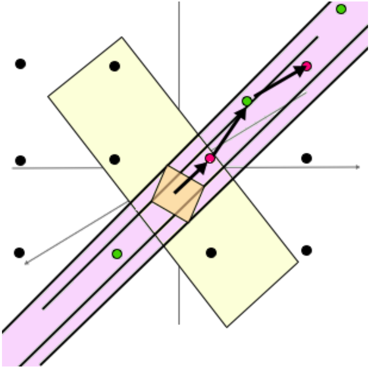
  $|\mathcal{E}| = N < \infty$

Rectilinear PET

via

Cut-and-Project Methods

Construction



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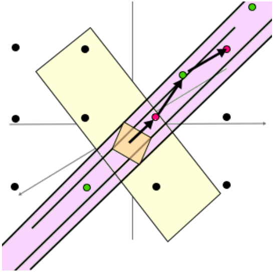

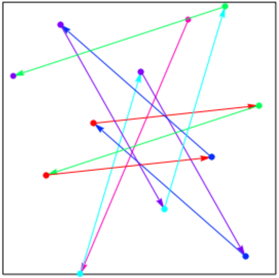

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- We define an order on the elements in $\mathcal{E}$ by

$$\eta_i < \eta_j \quad \text{if} \quad \pi_e(\eta_i) < \pi_e(\eta_j).$$

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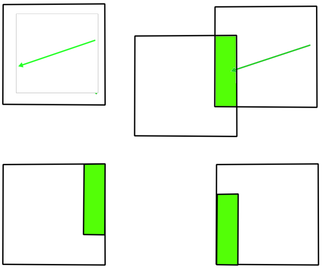

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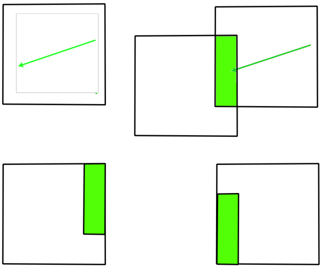

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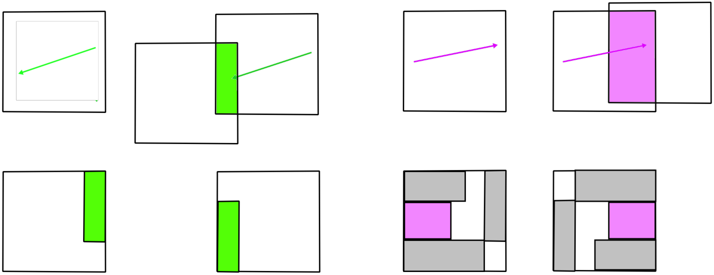

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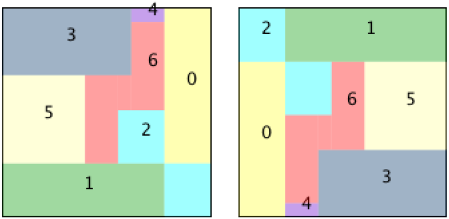
Construction

- An example of the rectilinear PET $T_n : X \to X$ constructed from cut-and-project method via matrix
$$M_n = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -n & n+1 \end{bmatrix} \text{ for } n = 6.$$



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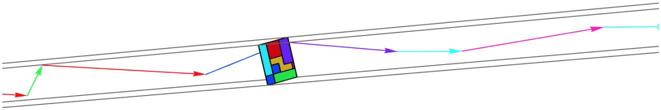
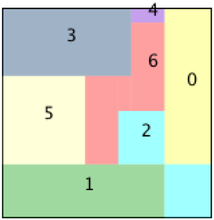
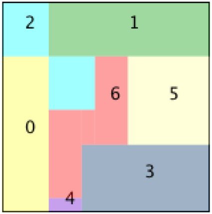
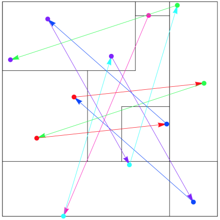
$$M_n = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -n & n+1 \end{bmatrix} \text{ for } n = 6.$$
- Each orbit of point $x \in X$ corresponds to a lattice walk.









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Let $Y \subset X$ be the rectangle with top right vertex $(1, 1)$ with width λ_1 and height λ_2 where $0 < \lambda_1 < \lambda_2 < 1$ are two eigenvalues of M_n. Let $T_n : X \to X$ be a rectilinear PET arisen from cut-and-project method via M_n. Then

$$\hat{T}_n|_Y = \psi^{-1} \circ T_n \circ \psi$$

Renormalization

Theorem

Let $Y \subset X$ be the rectangle with top right vertex $(1,1)$ with width λ_1 and height λ_2 where $0 < \lambda_1 < \lambda_2 < 1$ are two eigenvalues of M_n. Let $T_n : X \to X$ be a rectilinear PET arisen from cut-and-project method via M_n. Then

$$\hat{T}_n|_Y = \psi^{-1} \circ T_n \circ \psi$$

where $\psi : X \to X$ is defined by

$$\psi : (x,y) \mapsto (\frac{x+\lambda_1-1}{\lambda_1}, \frac{y+\lambda_2-1}{\lambda_2}).$$

Renormalization

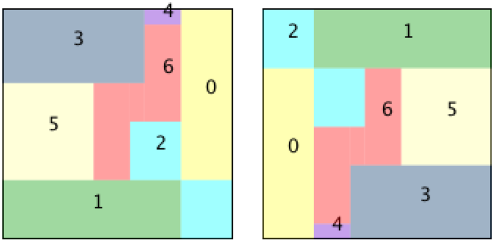
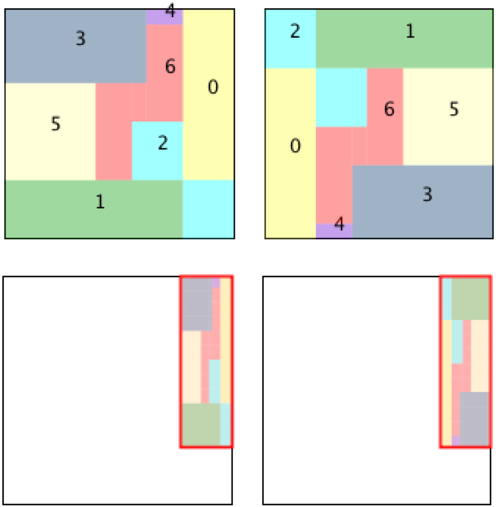


Figure: An illustration of renormalization.

Renormalization



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$$\Psi : \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -n \\ 0 & 1 & n+1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

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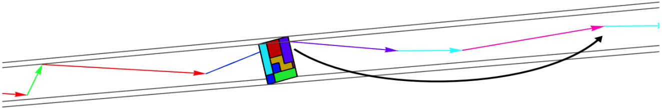
We show that $\Psi : \Lambda_Y \to \Lambda_Y$ corresponds to $\hat{T}|_Y : Y \to Y$.



Renormalization

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$$\pi_c\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (x, y) \quad \Rightarrow \quad \pi_c \circ \Psi\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (\lambda_1 x, \lambda_2 y)$$

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- First return

The map Ψ preserves the order of the lattice walk $\{\omega_1, \omega_2, \cdots\}$, i.e.

$$\pi_e(\omega_i) < \pi_e(\omega_j) \quad \Rightarrow \quad \pi_e \circ \Psi(\omega_i) < \pi_e \circ \Psi(\omega_j).$$

$\square$

Renormalization

Underlying substitution symbolic dynamics:

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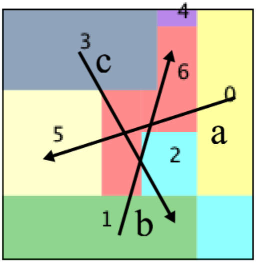
- Set $a = v_0$, $b = v_1$ and $c = v_3$.



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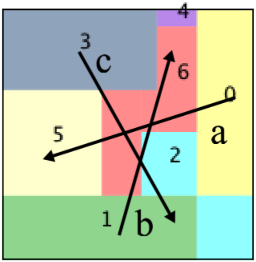

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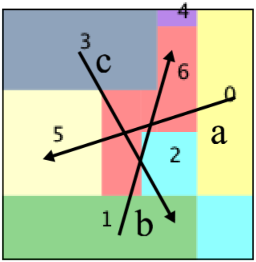

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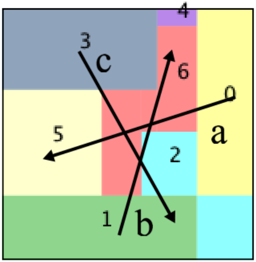

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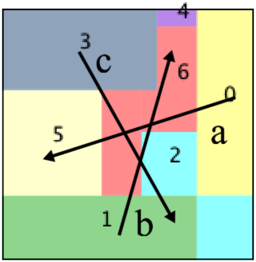

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- $W_n \sim M_n$

Multi-Stage Rectilinear PETs

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Let P_n be the matrix of translation. Consider matrix products

$$P_0 = P_{n_1} \cdots P_{n_k} \quad \text{and} \quad P_i = P_{n_1} \cdots P_{n_i} \quad \text{for } i \geq 1.$$

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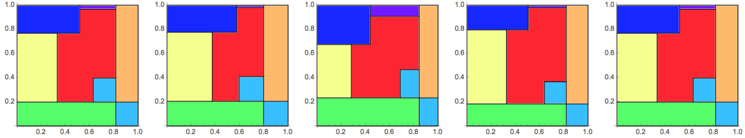

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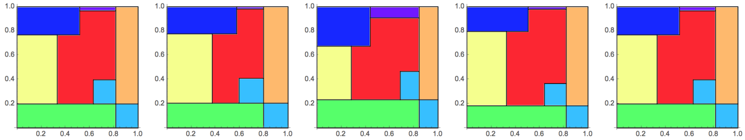
- each S_i has the same combinatorics as T_6.
- each S_i is renormalizable, i.e.

$$\hat{S}_i|_{Y_i} = \psi_i^{-1} \circ S_{i+1} \circ \psi_i.$$



Multi-Stage Rectilinear PETs

The matrix of translation

$$P_n = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & n-4 & n-3 & n-4 & n-4 \end{bmatrix}$$

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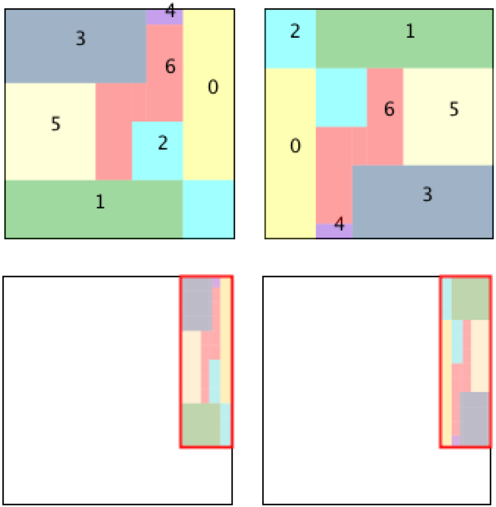
The matrix P_n can be reduced to the incidence matrix W_n.

Multi-Stage RETs (Construction)



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There is a RET $S : X \to X$ with the set of translation vectors $V = \{v_j\}_{j=0}^{6}$ if $v_j \in (-1, 1) \times (-1, 1)$.

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There is a **multi-stage RET** $S : X \to X$ if at each stage i, all translation vectors $v_j^i \in (-1,1) \times (-1,1)$.
A product P is called **admissible** if there is a multi-stage RET.

Multi-Stage RETs (Construction)

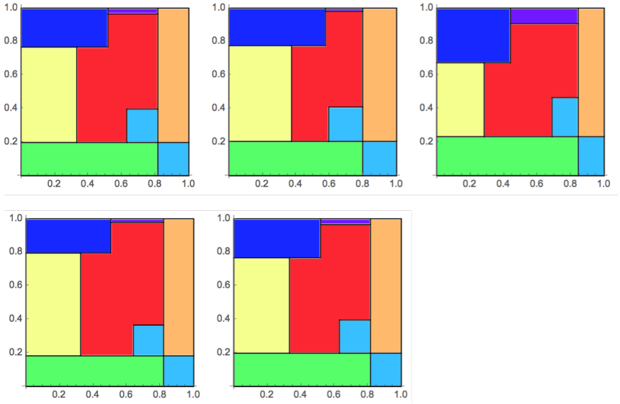

Figure: The multi-stage rectilinear PETs for $P = P_7 P_8 P_6 P_7$.

Multi-Stage RETs (Renormalization)

Theorem (Renormalization Scheme)

Let $P = P_{n_1} \cdots P_{n_k}$ be an admissible matrix product. Let $S : X \to X$ be a a multi-stage RET determined by P. Then

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The Parameter Space of Multi-Stage RETs

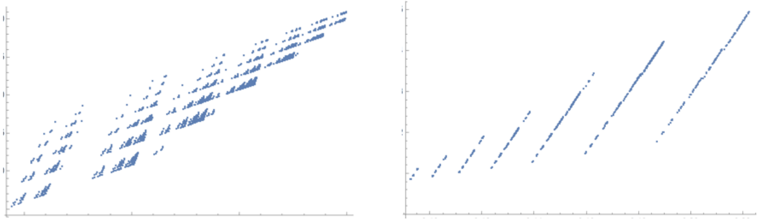

Figure: The parameter space of admissible rectilinear PETs in $\mathbb{R}^4$

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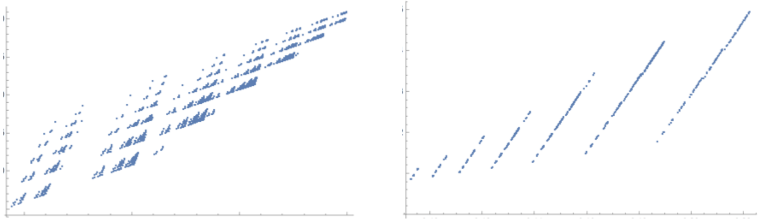

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Theorem (in process)

The parameter space $\mathcal{M}$ of admissible rectilinear PETs is a Cantor set in $\mathbb{R}^4$.

The Parameter Space of Multi-Stage RETs

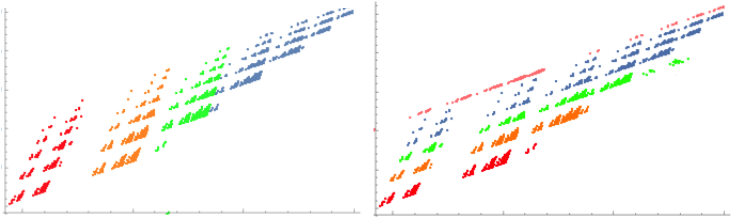
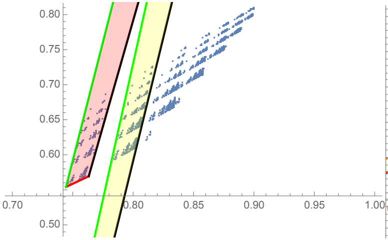
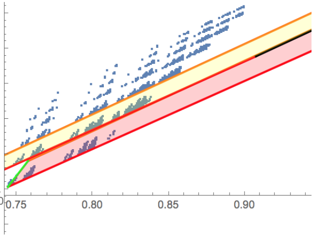


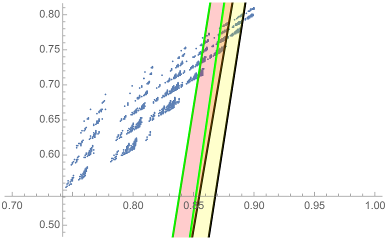
Figure: $P_n\mathcal{M}$ and $P_n^{-1}\mathcal{M}$

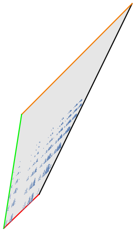
The Parameter Space of Multi-Stage RETs

We expect the dynamics on the parameter space to be a 'discrete horseshoe map'









Domain Exchange Transformation

We can construct domain exchange transformation on more general domains by the cut-and-project method.

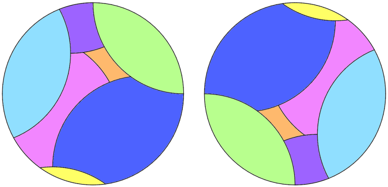


Figure: An example of circle exchange transformation by the cut-and-project method

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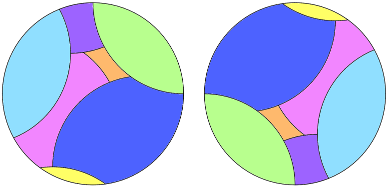

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(Conjecture) The domain exchange transformations on convex domains via cut-and-project methods are renormalizable.

Future Directions

- PETs on general domains and their renormalizations
- Piecewise isometries arisen from cut-and-project methods associated to quartic polynomials
- Generalizations of the Three Gap Theorem
- Self-similar tilings from RETs?
- Generalizations of Rauzy inductions in PETs
- Complexity of the PETs

Thank you