Chaire Jean-Morlet -- Akiyama-Arnoux Semester--Small Group Conference on Zero-Entropy Systems
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Global Scaling in a One-Parameter Family of Parametric Kicked-Oscillator Maps

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Review

A one-dimensional harmonic oscillator is kicked four times per natural period. Each kick is an instantaneous momentum shift proportional to a periodic, piecewise linear function of the position.

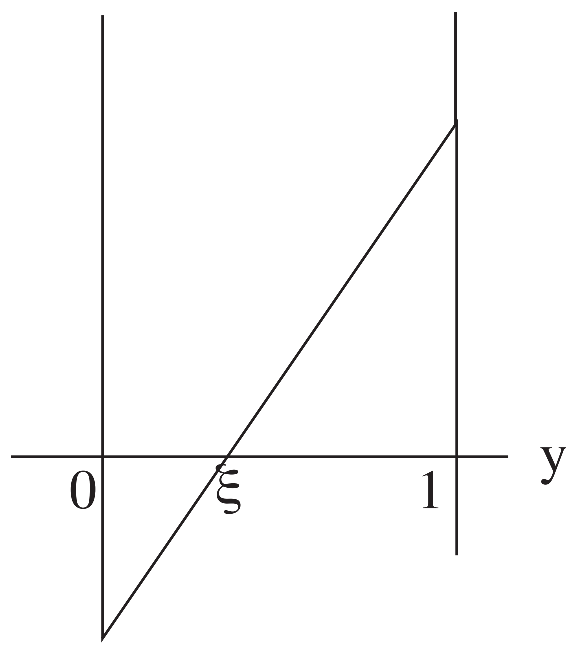
We consider the Poincaré map on the 2D phase space, which has the periodicity of a 2D square crystal.

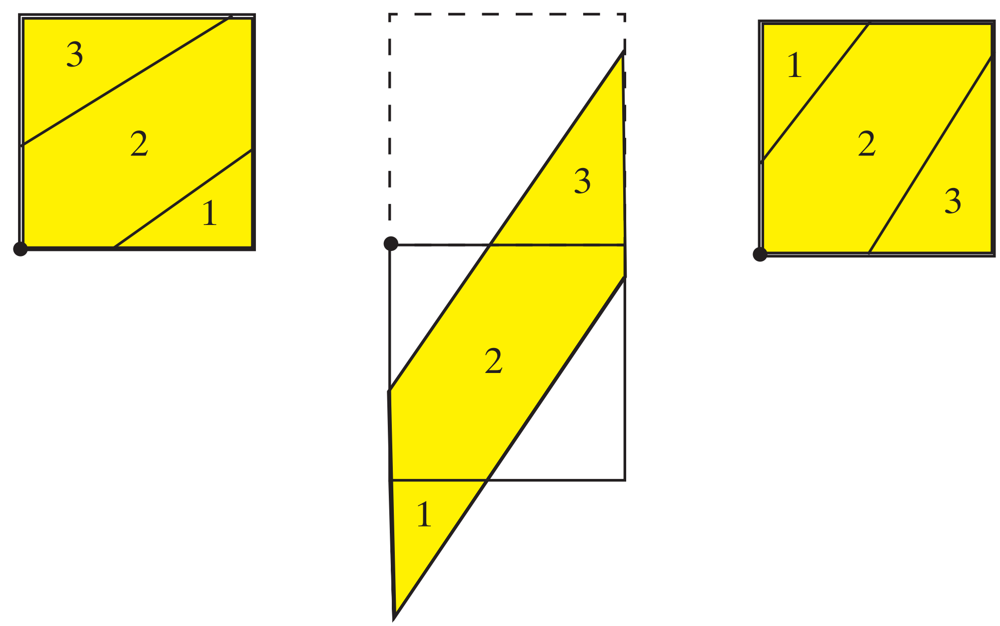
Are there aperiodic orbits which tend to infinity for asymptotically long times, with non-diffusive power-law behavior?

Global Kicked-Oscillator Map

$$W\begin{pmatrix} x \\ y \end{pmatrix} = R(-\pi/2)\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ f(y) \end{pmatrix} = \begin{pmatrix} y \\ -x + f(y) \end{pmatrix}$$

$$f(y) = \alpha\,(\,y \bmod 1 - \xi\,)$$



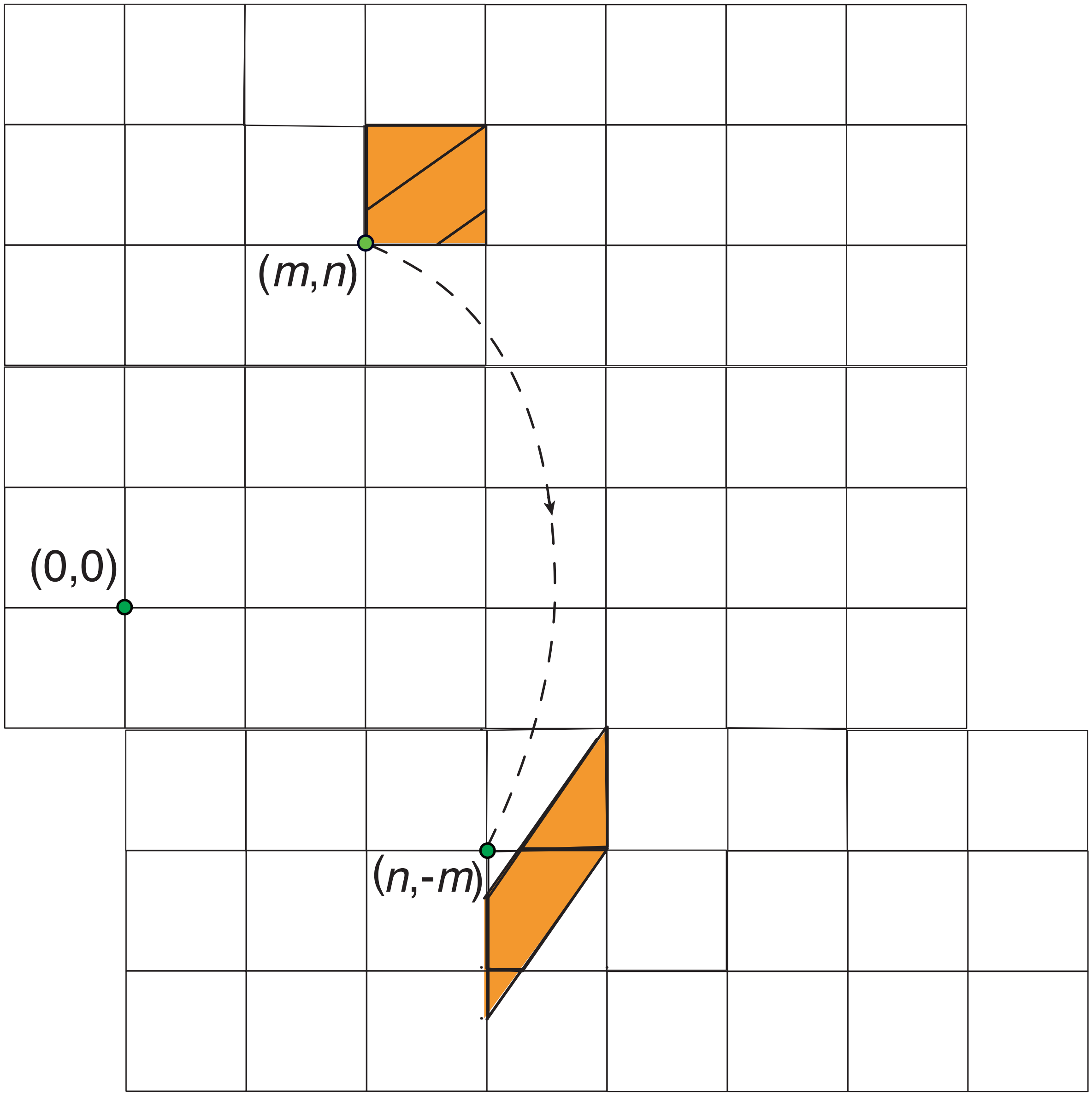


Local Map K

$$K\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ (-x + \alpha\, y - \xi) \bmod 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 3-\iota(x,y) \end{pmatrix}$$

piecewise affine transformation

Action of map W on the plane

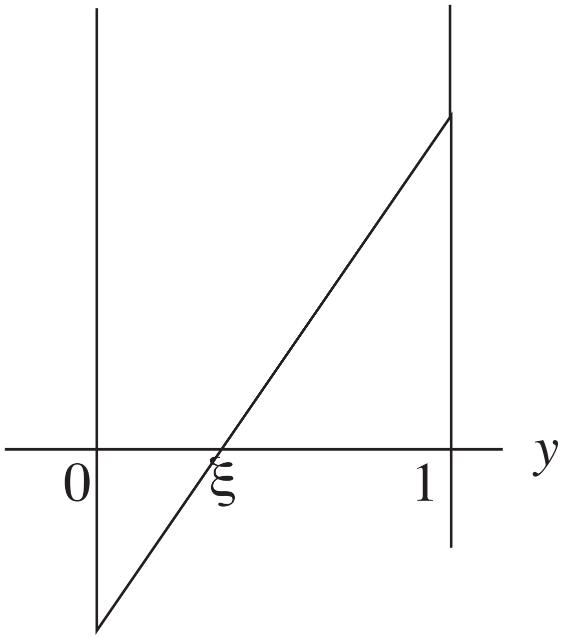


W decomposes into K plus a Z^2 lattice isometry

$$W(\boldsymbol{u} + \boldsymbol{m}) = K(\boldsymbol{u}) + R(-\pi/2)\,\boldsymbol{m} + d(\boldsymbol{u})$$

$$d(\boldsymbol{u}) = \begin{cases} (0,-2) & \boldsymbol{u} \in D_1 \\ (0,-1) & \boldsymbol{u} \in D_2 \\ (0,0) & \boldsymbol{u} \in D_3 \end{cases}$$

$$f(y) \;=\; \alpha\,(\,y \bmod 1 - \xi\,)$$



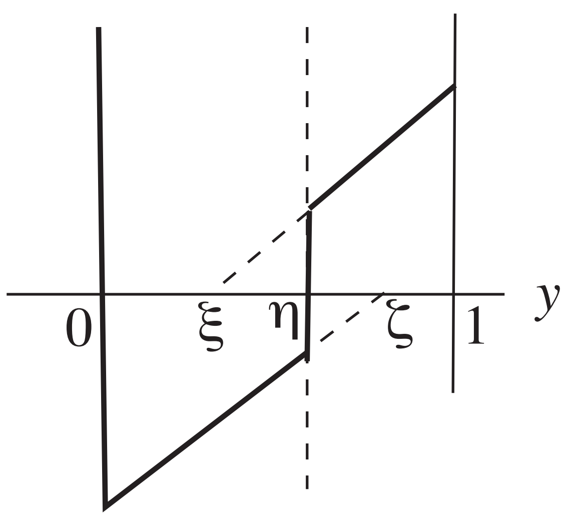
My choice of parameters:

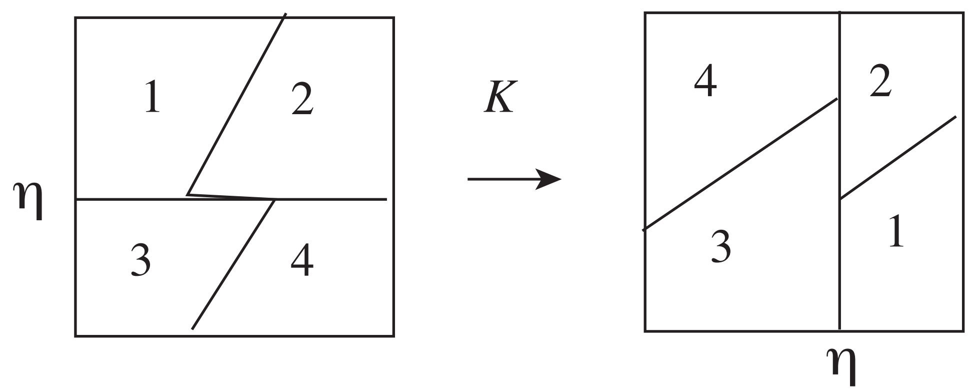
slope α : $\sqrt{2}$

intercept $\xi = \frac{\beta}{2}(\alpha - \beta\, s\,)$, $\quad 0 \leqslant s < \alpha$

Notation:
$\beta = \alpha - 1$
$\omega = \alpha + 1$

More complicated $f(y)$:

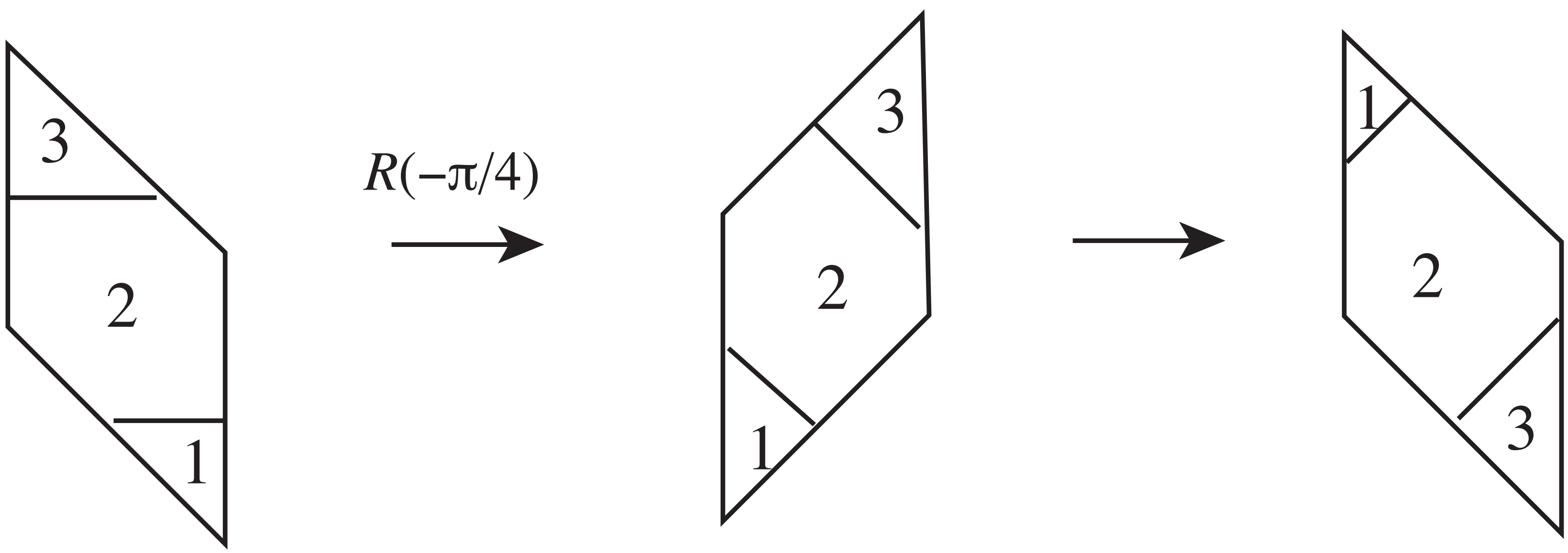


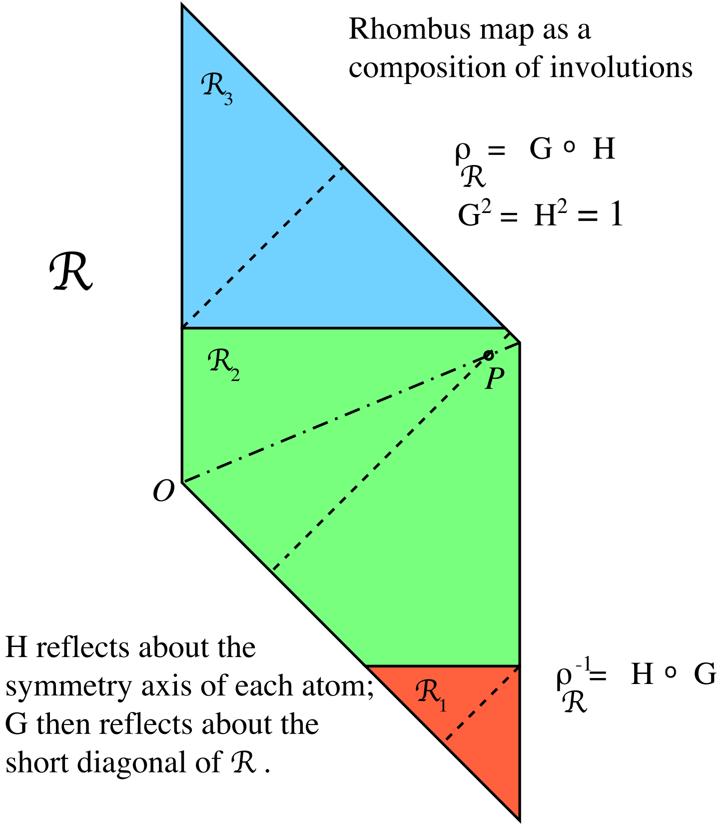


K expressed as a true piecewise isometry

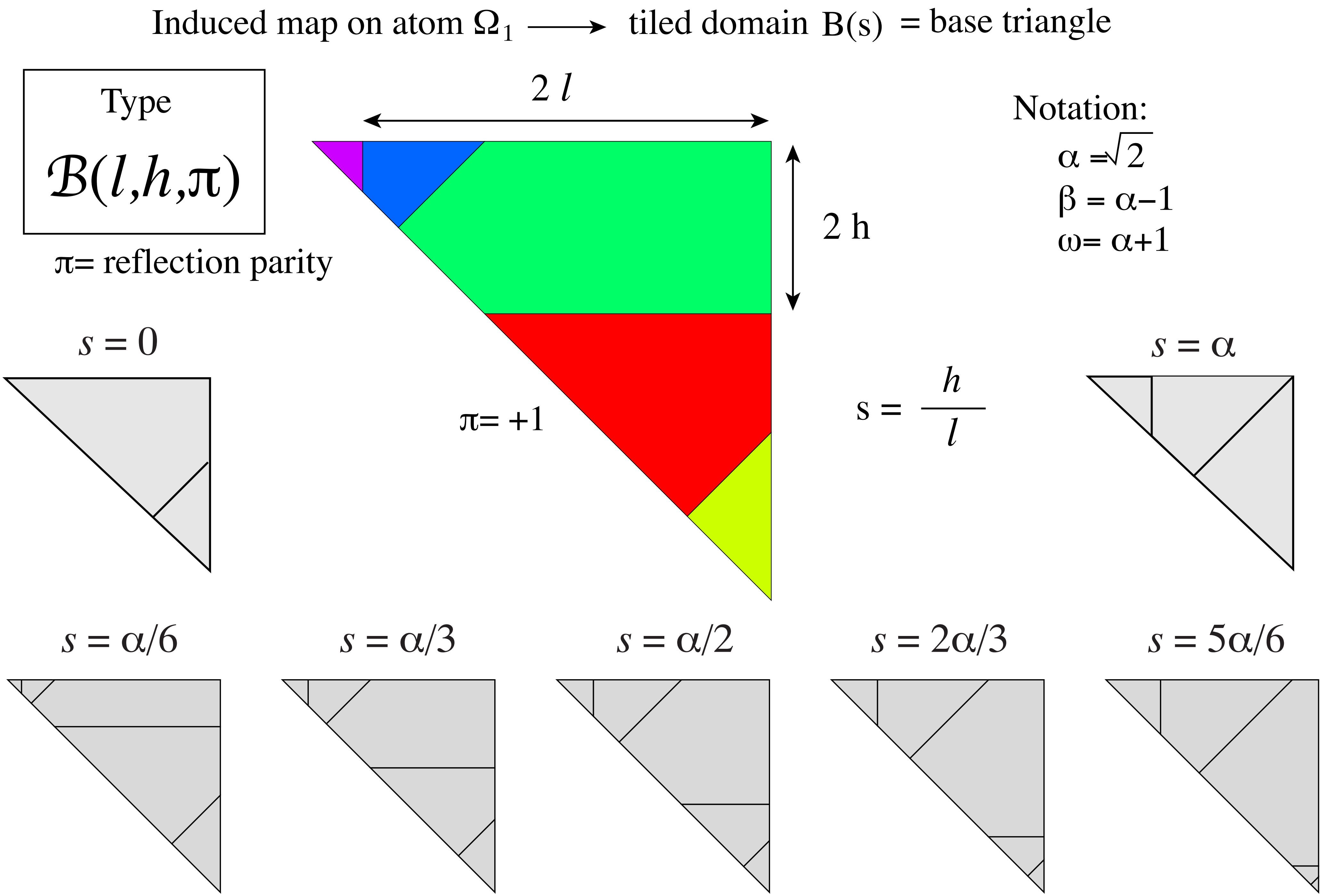
K is conjugate to a rotation followed by a piecewise translation, applied to a rhombus with vertex angle θ .

Example for $\alpha = \sqrt{2}$, $\theta = \pi/4$





Rhombus map as a composition of involutions
$\rho_{\mathcal{R}} = G \circ H$
$G^2 = H^2 = 1$
$\mathcal{R}$
$\mathcal{R}_3$
$\mathcal{R}_2$
P
O
$\mathcal{R}_1$
$\rho_{\mathcal{R}}^{-1} = H \circ G$
H reflects about the symmetry axis of each atom; G then reflects about the short diagonal of $\mathcal{R}$.


Induced map on atom Ω_1 ⟶ tiled domain B(s) = base triangle
Type
$\mathcal{B}(l,h,\pi)$
π= reflection parity
2 l
2 h
Notation:
$\alpha = \sqrt{2}$
$\beta = \alpha - 1$
$\omega = \alpha + 1$
$s = 0$
π= +1
$s = \frac{h}{l}$
$s = \alpha$
$s = \alpha/6$
$s = \alpha/3$
$s = \alpha/2$
$s = 2\alpha/3$
$s = 5\alpha/6$

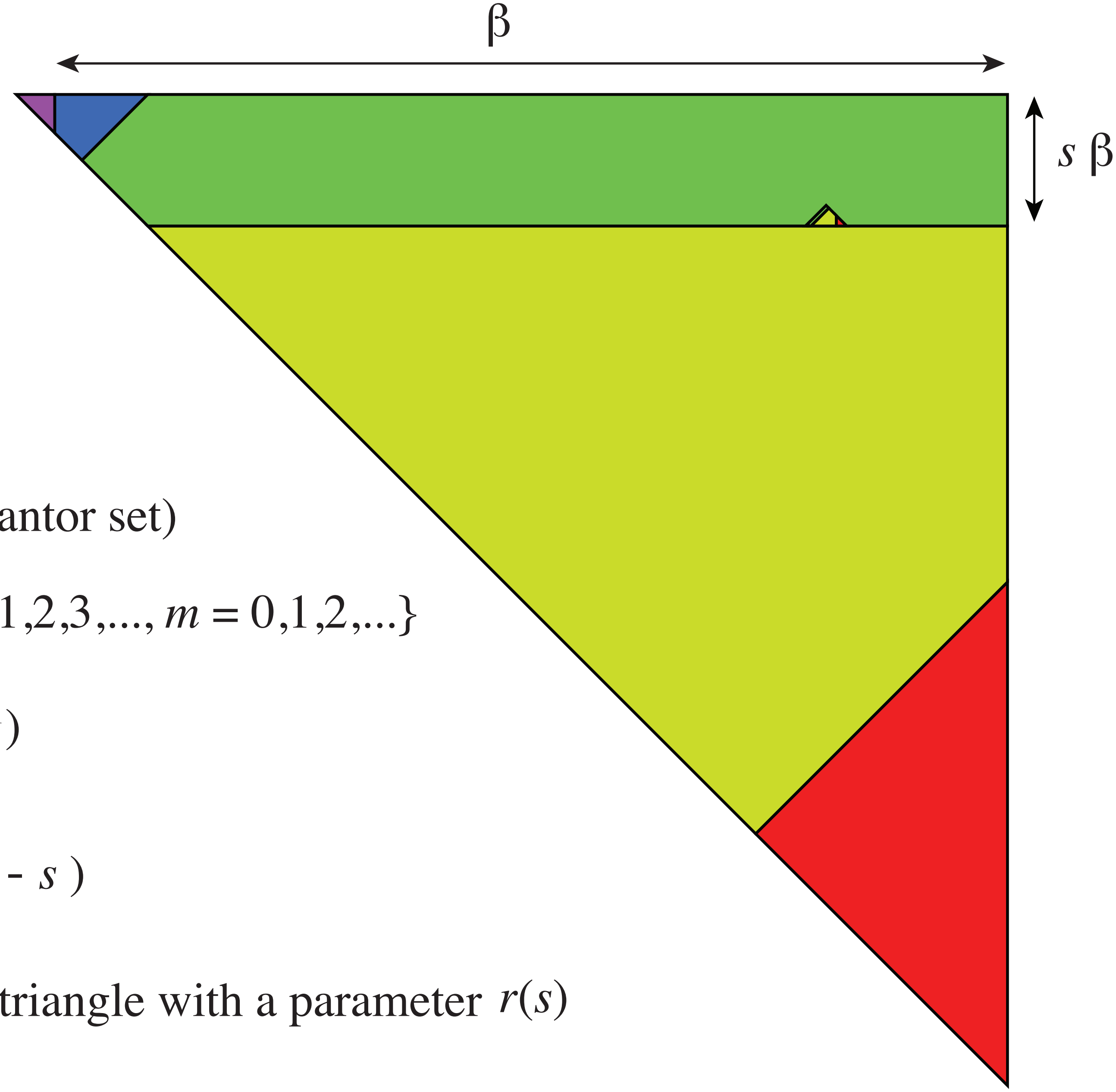
Notation

$\alpha = \sqrt{2}$

$\beta = \alpha - 1$

$\omega = \alpha + 1$

Renormalization for a special parameter set



Special parameter set (Cantor set)

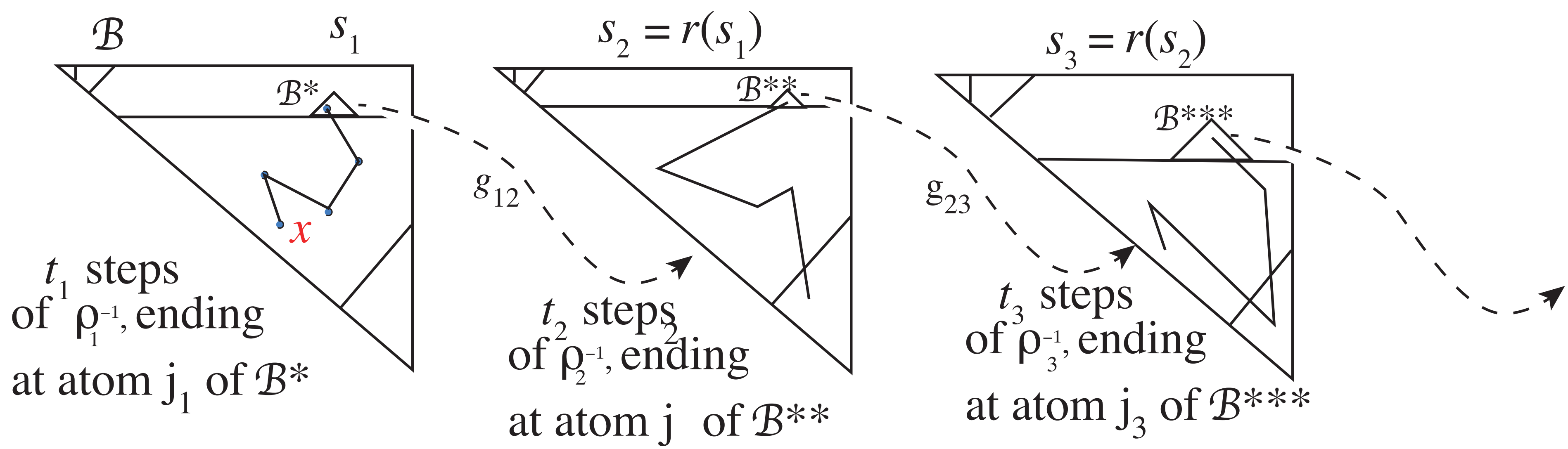
$$\hat{S} = \{ s : r^m(s)\ \varepsilon\ I_{-2k,0},\ \ k = 1,2,3,\ldots,\ m = 0,1,2,\ldots\}$$

$$I_{-2k,0} = (\alpha\,\beta^{2k+1}, 2\,\beta^{2k+1})$$

$$r(s) = \omega^{2k+2}(2\,\beta^{2k+1} - s)$$

Small base triangle is a base triangle with a parameter $r(s)$ "pencil".

Symbolic representation of boundary-avoiding aperiodic points (residual set)



Point x corresponds to code

$$\begin{pmatrix} s_1 \\ j_1 \\ t_1 \end{pmatrix}\begin{pmatrix} s_2 \\ j_2 \\ t_2 \end{pmatrix}\begin{pmatrix} s_3 \\ j_3 \\ t_3 \end{pmatrix}\begin{pmatrix} s_4 \\ j_4 \\ t_4 \end{pmatrix} \cdot \cdot \cdot \cdot$$

For an n-cycle point

$$x = g_{n-1,n} (\rho_{n-1}^{-1})^{t_{n-1}} \cdots\cdots g_{2,3} (\rho_2^{-1})^{t_2} g_{1,2} (\rho_1^{-1})^{t_1} \; x$$

$$x = S \circ R\, x + \delta \qquad (\, S = \text{scaling}, R = \text{rotation matrix})$$

$$x = (1 - S \circ R)^{-1} \delta$$

Rules for Symbolic Updating

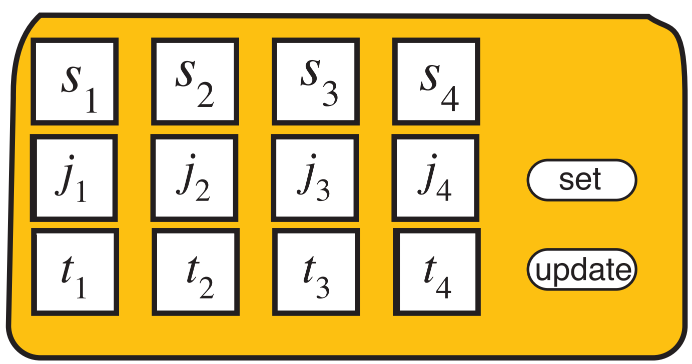
$$\left(\begin{pmatrix} s_1\\ j_1\\ t_1 < t_1^{max}\end{pmatrix} \;\dots\right) \longmapsto \left(\begin{pmatrix} s_1\\ j_1\\ t_1+1\end{pmatrix}\;\dots\right)$$

$$\left(\begin{pmatrix}s_1\\j_1\\t_1^{max}\end{pmatrix}\begin{pmatrix}s_2\\j_2\\t_2^{max}\end{pmatrix}\dots\begin{pmatrix}s_k\\j_k\\t_k^{max}\end{pmatrix}\begin{pmatrix}s_{k+1}\\j_{k+1}\\t_{k+1}<t_{k+1}^{max}\end{pmatrix}\dots\right)\longmapsto$$

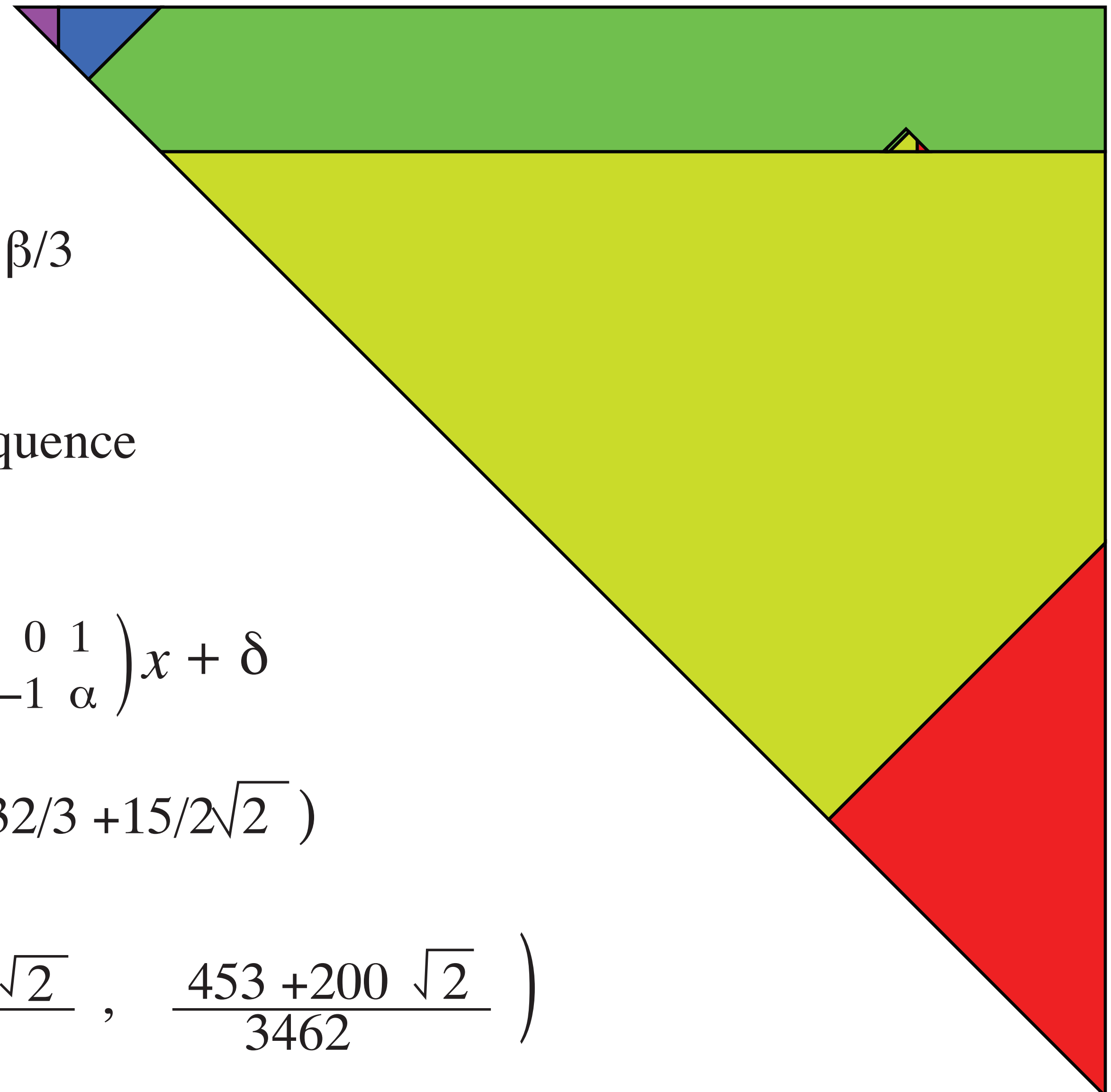
$$\left(\begin{pmatrix}s_1\\j'_0\\0\end{pmatrix}\begin{pmatrix}s_2\\j'_2\\0\end{pmatrix}\dots\begin{pmatrix}s_k\\j'_k\\0\end{pmatrix}\begin{pmatrix}s_{k+1}\\j_{k+1}\\t_{k+1}+1\end{pmatrix}\dots\right)$$

$$j'_i = p(s_{i+1}, j_{i+1}, 0),\quad i = 1, \dots k \qquad , \qquad j_k = p(s_{i+1}, j_{k+1}, t_{k+1}+1)$$

The codometer



Simple example of symbolic coding



$s^* \;=\;$ fixed point of $r(s)$ in $I_{-2,0} \;=\; \beta/3$

$$x = \begin{pmatrix} s^* \\ 3 \\ 0 \end{pmatrix}^{\infty}$$

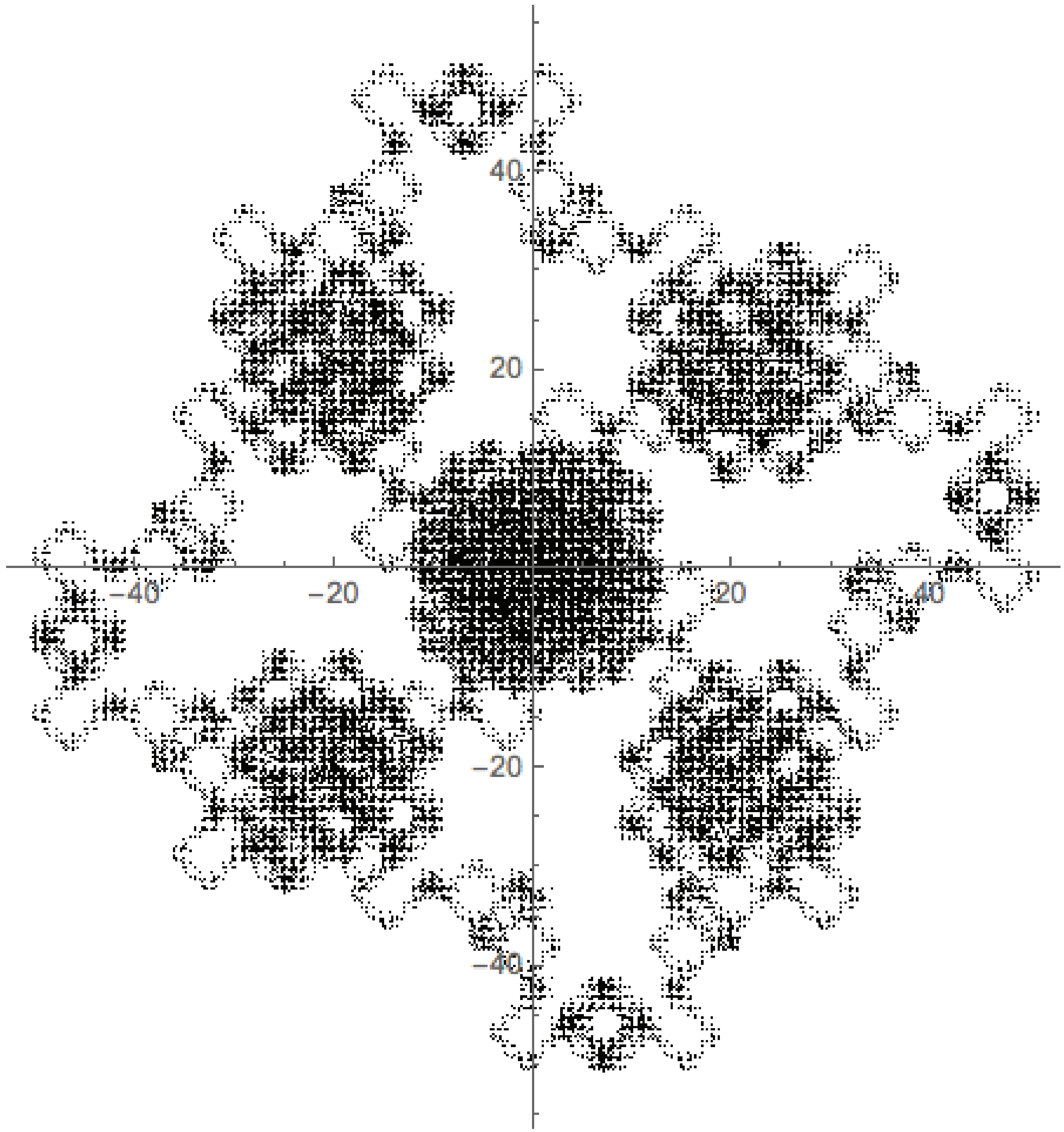
period-1 code sequence

$$x = \; g_{12}(x) \; = \omega^4 \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix} x + \delta$$

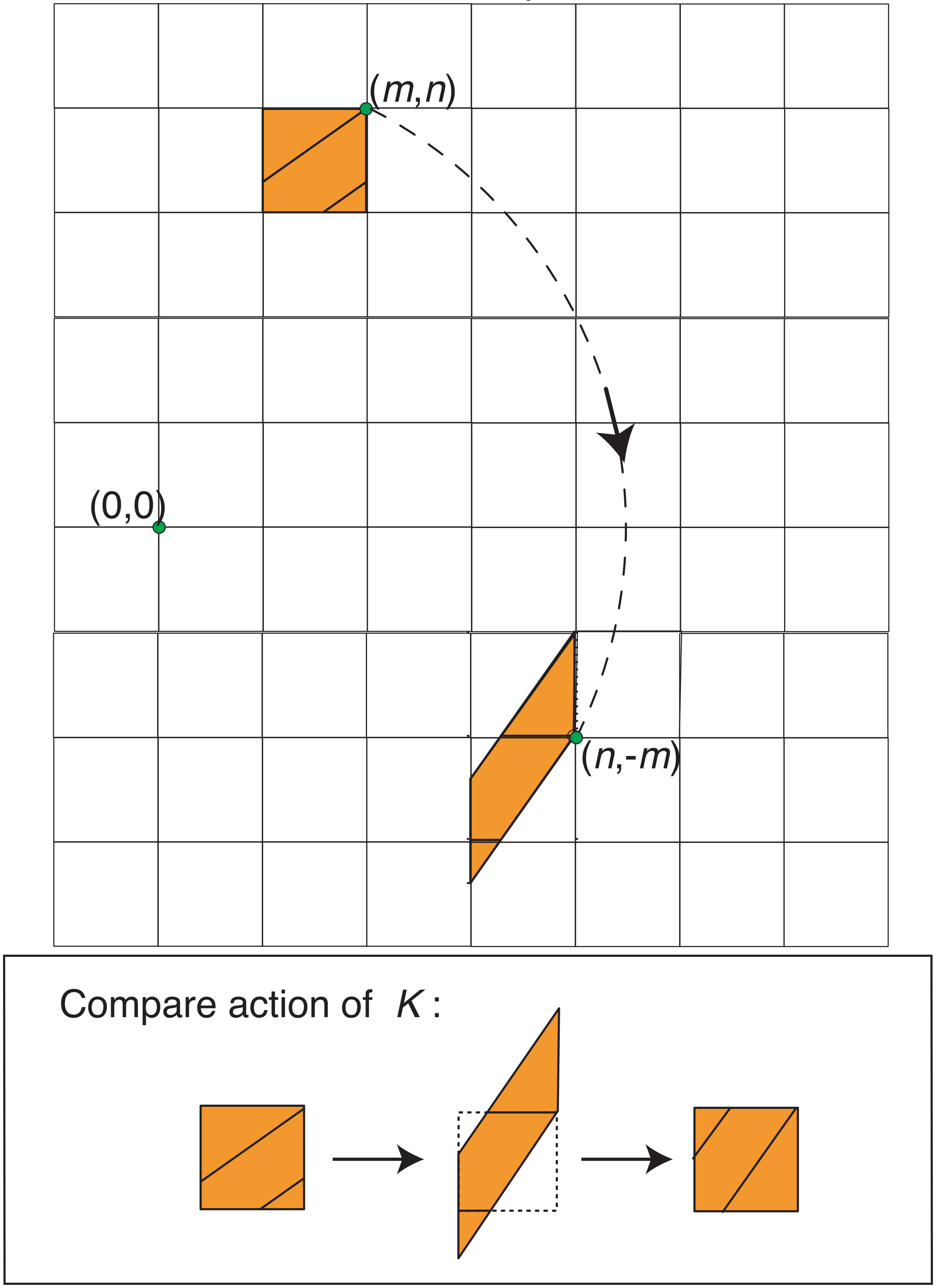
$$\delta = \left(-3-7/3\sqrt{2},\; 32/3 + 15/2\sqrt{2}\right)$$

$$x = \left(\frac{2115 + 758\sqrt{2}}{3462},\quad \frac{453 + 200\sqrt{2}}{3462}\right)$$

100000 iterations of the global map, starting at x



Action of map W

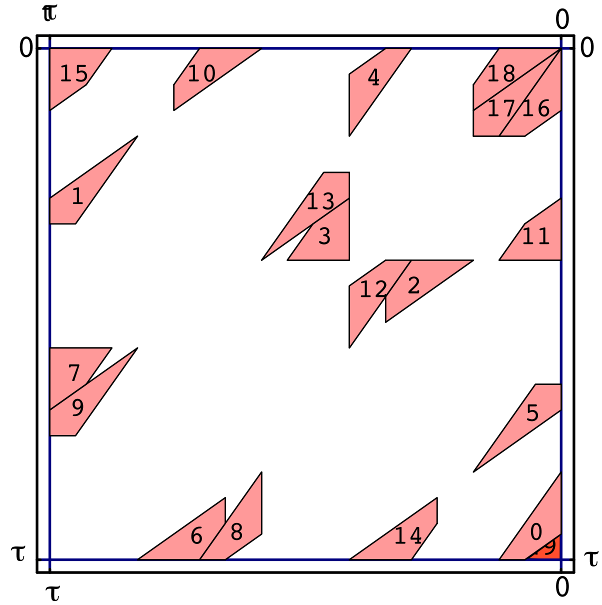


$$K u = C u - d(u)\tau \qquad F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix}$$

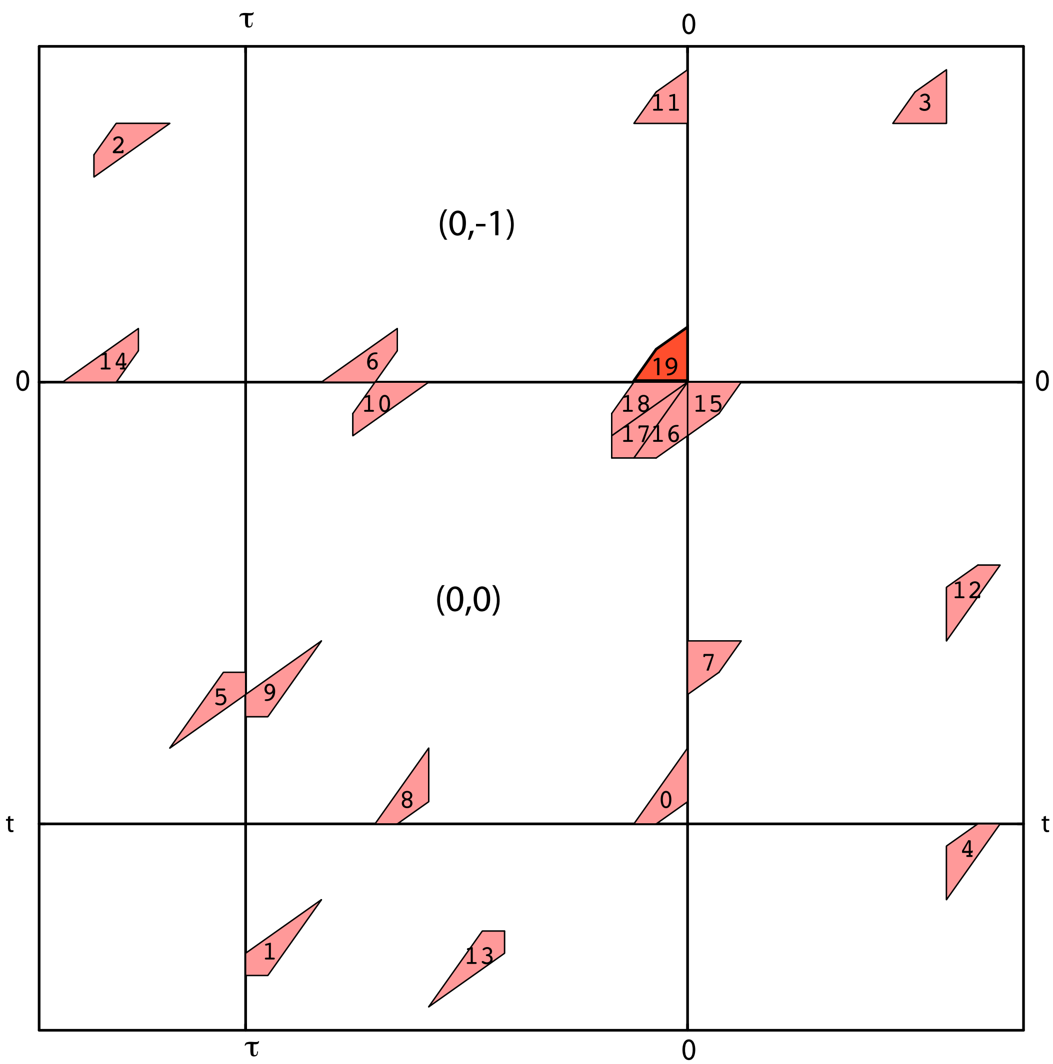
$$W[u, z] = [K u, \; F z + d(u)] \qquad d(u) = d_j, \quad u \in \Omega_j, \quad z \in Z^2$$

Tiles

$D_0(0) = D_0^{\,0}(0)$,
$D_0^{\,1}(0),\ldots\ldots,D_0^{\,18}(0)$



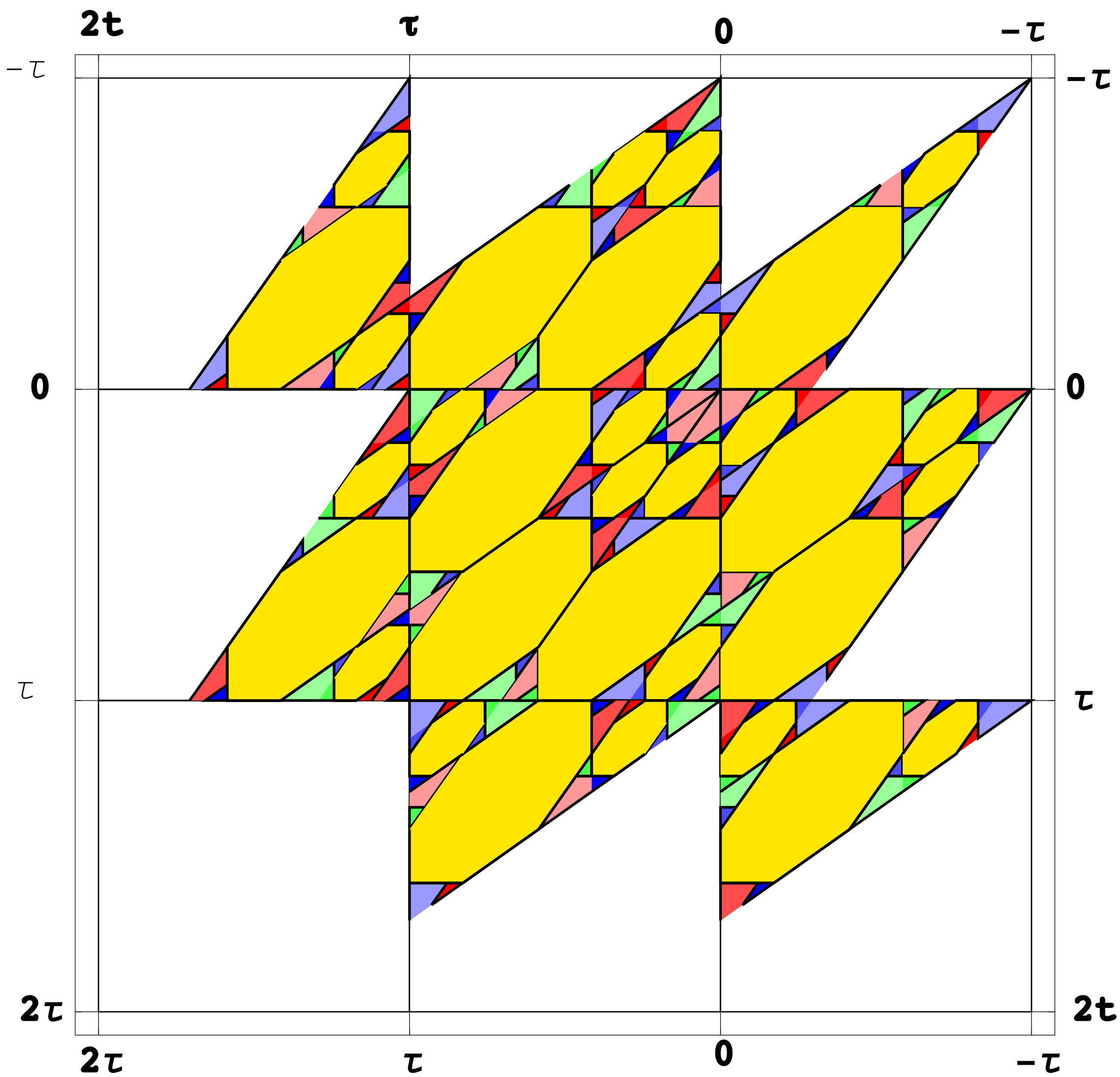
K - orbit of $D_0(0)$ in Ω

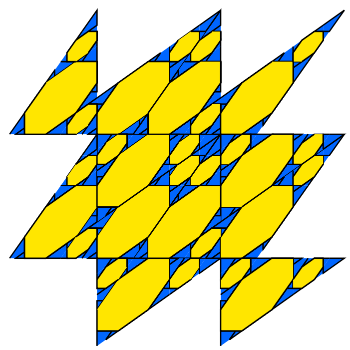


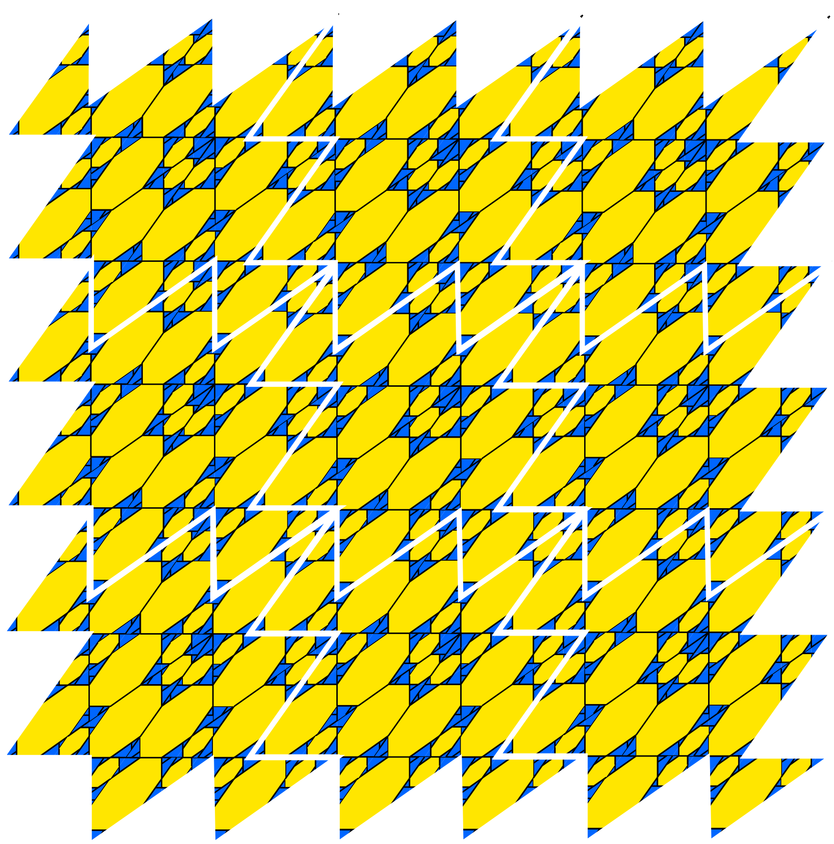
W - orbit of $D_0(0)$ in $\boldsymbol{R}^2 = \Omega \times \boldsymbol{Z}^2$

Tiles [$D_0^{\,0}(0)$,(0,0)], [$D_0^{\,1}(0)$, (0,1)] , [$D_0^{\,2}(0)$,(1,-1)], . . . , [$D_0^{\,18}(0)$, (0,0)]

Combining orbits initiated in cells [$D(0)$, (0,0)] , [$D(0)$, (0,1)] , [$D(0)$, (1,1)] , and [$D(0)$, (1,0)] with the lowest-level periodic orbits (yellow octagons) produces a ***supertile***. These invariant non-convex polygonal regions tile the plane with periodicity 2τ in both *x* and *y* directions. Here $t = -\sqrt{2}/2$.







Scaling for the lifted map : a third scale factor

Represent $\mathbf{R}^2$ as

$$\Omega \times \mathbf{Z}^2 = \{ [(x,y),(m,n)] : (x,y) \in \Omega, (m,n) \in \mathbf{Z}^2 \}$$

Then, for $u \in \Omega_j$, $\boldsymbol{m} \in \mathbf{Z}^2$,

$$F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\boldsymbol{m} = (m,n)$

$$W[u,\boldsymbol{m}] = [Ku, F\boldsymbol{m} + \Delta_j]$$

$\Delta j = (0, dj)$

Complex representation

$$W[u,\zeta] = [Ku, i\zeta + d_j]$$

$\zeta = n + i\,m$

For the return maps, $u \in D_j(L+1)$, $z \in \mathbf{Z}^2$,

$$\rho_W(L+1)[u,\zeta] = [\rho_K(L+1)u, i^{T_j(L+1)}\zeta + d_j(L+1)],$$

where

$$d_j(L+1) = i^{\kappa(j,0)} d_{p(j,0)}(L) + i^{\kappa(j,1)} d_{p(j,1)}(L) + \cdots\cdots + d_{p(j,\nu_j-1)}(L)$$

$$\kappa(j,t) = \sum_{k=t+1}^{\nu_j-1} T_{p(j,k)}$$

$$d_j(L+1) = \sum_{k=0}^{J-1} M_{jk}\, d_k(L)$$

M_{jk} = matrix of Gaussian integers

Complex Representation

In place of $\Omega \times \mathbb{Z}^2$, we may represent R^2 as $\Omega \times \mathbb{C}$, $\mathbb{C}$ = complex plane. The formulas of the previous slide remain valid with the replacements

$$d_j(L) = (m,n) \in \mathbb{Z}^2 \longleftrightarrow n + i\,m \qquad (\,i = \sqrt{-1}\,)$$

$$F^p\,(m,n) \longleftrightarrow i^{\,p}\,(n + i\,m)$$

Global recursion matrix $\boldsymbol{M}$ has entries which are Gaussian integers (Re and Im parts are integers). The asymptotic scaling properties, for $L \longrightarrow \infty$, of the translation vectors $d_j(L)$ are revealed by examining the Jordan canonical form $\boldsymbol{J}(\boldsymbol{M})$ of $\boldsymbol{M}$.

The global geometric scale factor ω_W is defined as the absolute value of the largest-magnitude diagonal element of $\boldsymbol{J}(\boldsymbol{M})$.

Global Expansivity

A scaling sequence with translation vectors $d_j(L)$ is globally expansive if $\omega_W>1$,and there exists a complex vector ξ such that for $L \longrightarrow \infty$,

$$\boldsymbol{d}(L) \sim \omega_W^{\boldsymbol{L}}\,\xi$$

Long-Time Asymptotic Behavior of Residual Set Orbits

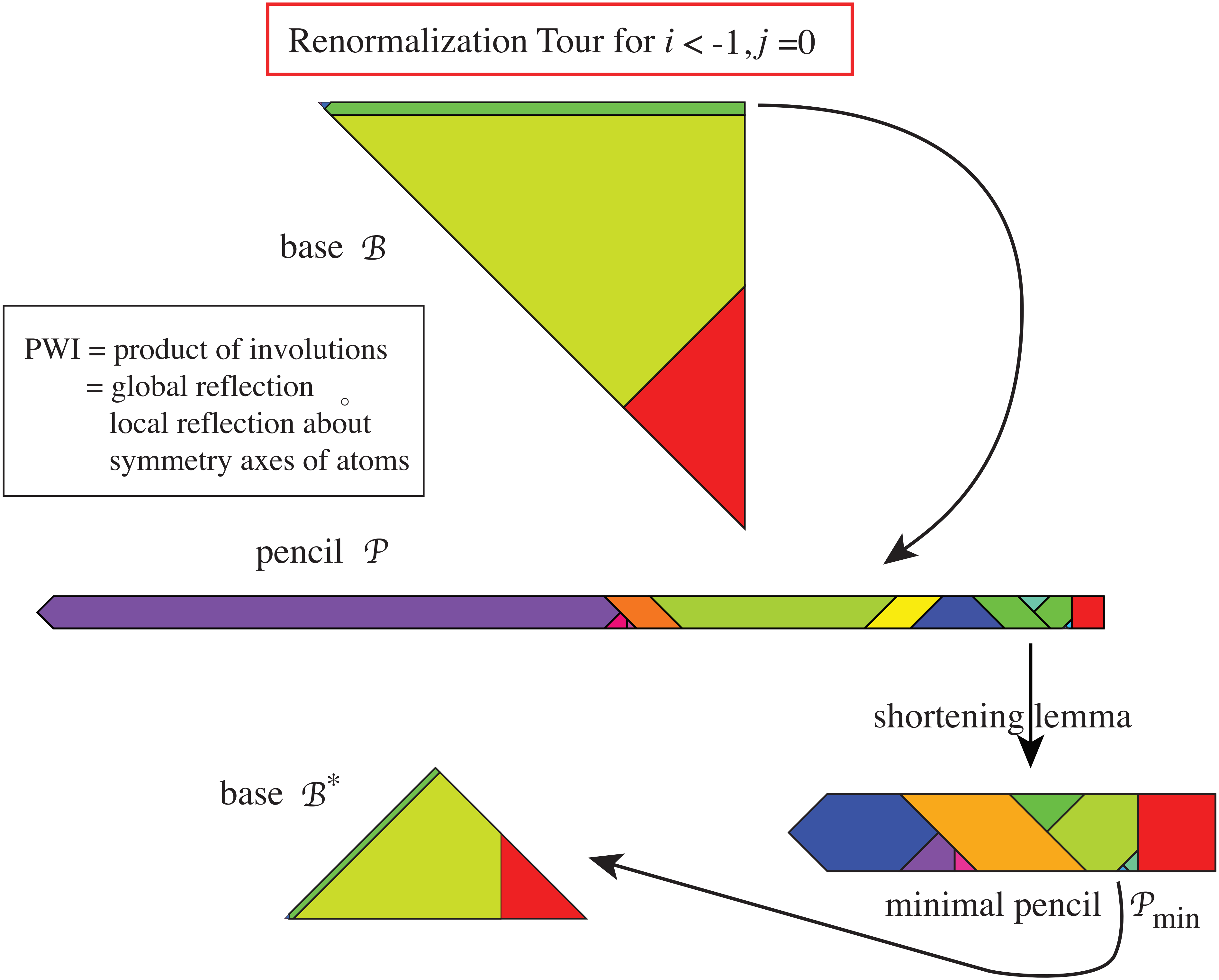
Theorem: Let z be a point in the residual set of a globally expansive scaling sequence, with temporal scale factor ω_T and global expansion factor ω_W.

Let

$$\mu = \frac{\log \omega_W}{\log \omega_T}$$

Then

$$\limsup_{t \to \infty} \frac{|W^t(z)|}{t^{\mu}} > 0$$


Renormalization Tour for $i < -1, j = 0$
base $\mathcal{B}$
PWI = product of involutions
= global reflection $\circ$
local reflection about
symmetry axes of atoms
pencil $\mathcal{P}$
shortening lemma
base $\mathcal{B}^*$
minimal pencil $\mathcal{P}_{\min}$

Renormalization scenario $\mathcal{R} \to \mathcal{B} \to \mathcal{P} \to \mathcal{P}_{\min} \to \mathcal{B}^*$ on the parameter intervals

$$r^m(s) \;\varepsilon\; I_{-2k,0} = (\alpha\, \beta^{2k+1}, 2\, \beta^{2k+1}), \qquad k = 1, 2, 3, \ldots, \; m = 0, 1, 2, \ldots$$

This is an r - invariant Cantor set whose Hausdorff dimension is determined by the equation

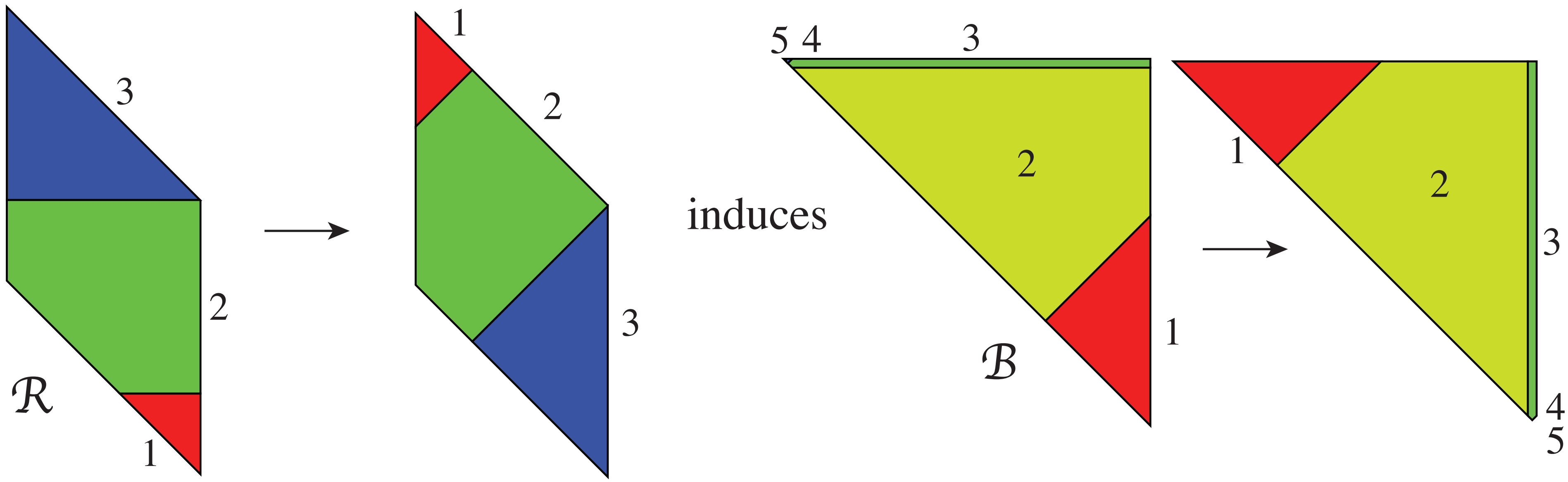
$$\sum_{k=0}^{\infty} \beta^{(2k+4)\,d} = \frac{\beta^{4\,d}}{1 - \beta^{2\,d}} = 1$$

$$d = \frac{\log \gamma}{2 \log \beta} \quad , \qquad \gamma = (\sqrt{5} - 1)/2$$

$$d = 0.2729897\ldots$$

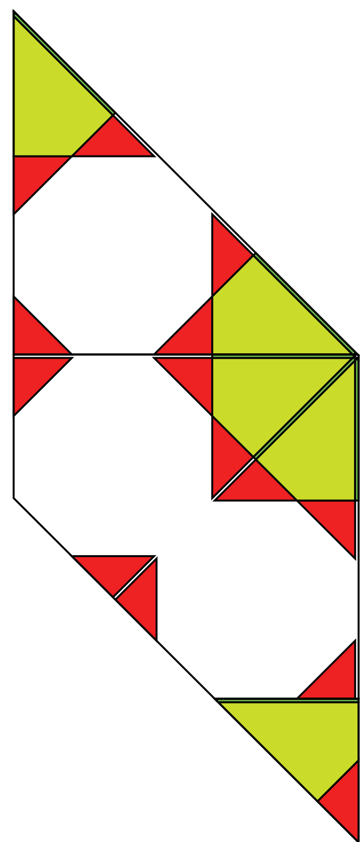
On $I_{-2k,0}$, $\quad r(s) = \omega^{2\kappa+2}\,(2\,\beta^{2\kappa+1} - s) \qquad \omega = \beta^{-1} = \alpha + 1$

Renormalization scenario for $s \in I_{-2k,\,0}$, step 1 : $\mathcal{R}$ induces $\mathcal{B}$



Return paths:

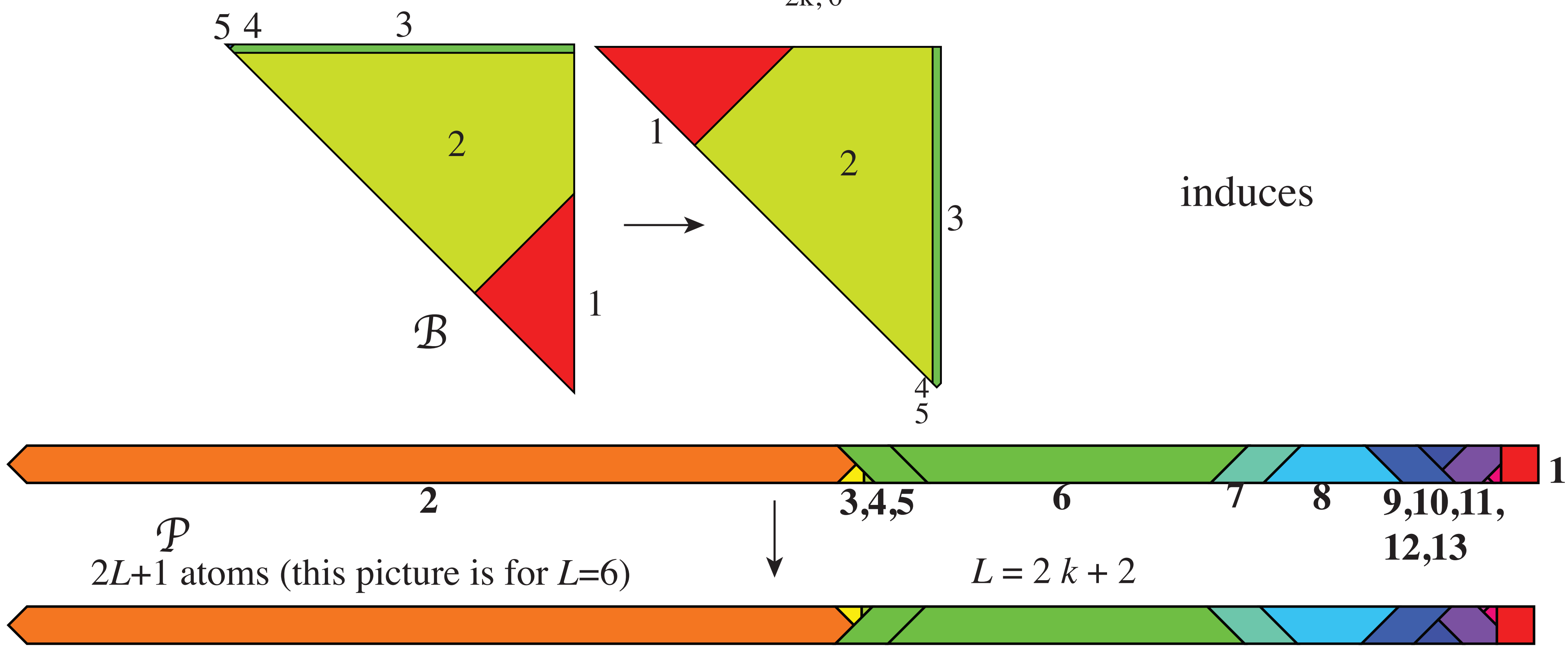
1: $1, (3, 2^2)^4, 3$
2: $1, 3, 2^2, 3$
3: $1, 3, 2^3, 3$
4: $1, 3, 2^4, 3$
5: $1, 3, 2^5, 3$



Incidence matrix

$$\begin{pmatrix} 1 & 8 & 5 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 4 & 2 \\ 1 & 5 & 2 \end{pmatrix}$$

Renormalization scenario for $s \in I_{-2k,0}$, step 2 : $\mathcal{B}$ induces $\mathcal{P}$



Return paths:

1: 3

2: 3, 1

3: 3, 1, 4, 1

4: 3, 1, 5, 1

5: 3, 2

6: $3, 2^6$

7: $3, 2^3$

8: $3, 2^2, \phi(1^5), 2$

9: $3, 2^2, \phi(1^2), 2$

$m = 5, 6, \ldots, L$

$2m$: $3, 2^2, \phi\big(1^2, \sigma(1)^2, \sigma^2(1)^2 \cdots, \sigma^{L-5}(1)^2, \sigma^{L-4}(1)^5\big), 2$

$2m+1$: $3, 2^2, \phi\big(1^2, \sigma(1)^2, \sigma^2(1)^2 \cdots, \sigma^{L-5}(1)^2, \sigma^{L-4}(1)^2\big), 2$

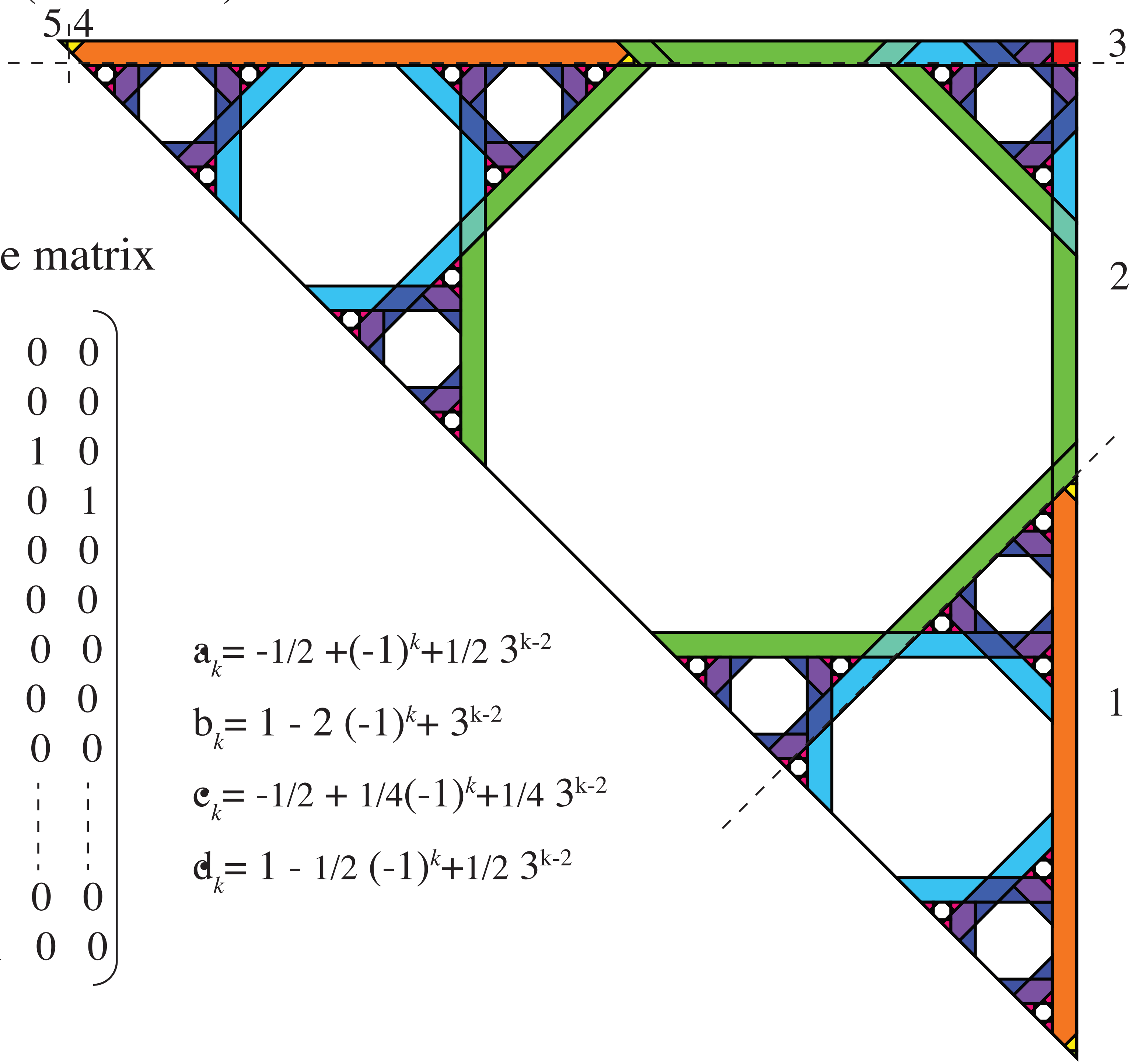
Substitutions: $\phi(1) = (1, 2)$

$\phi(2) = (2, 2, 2)$

$\sigma(1) = (2,1,1)$

$\sigma(2) = (1,1,1)$

$\mathcal{B} \to \mathcal{P}$ (continued)



Incidence matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ a_3 & c_3 & 1 & 0 & 0 \\ b_3 & d_3 & 1 & 0 & 0 \\ a_4 & c_4 & 1 & 0 & 0 \\ b_4 & d_4 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_L & c_L & 1 & 0 & 0 \\ b_L & d_L & 1 & 0 & 0 \end{pmatrix}$$

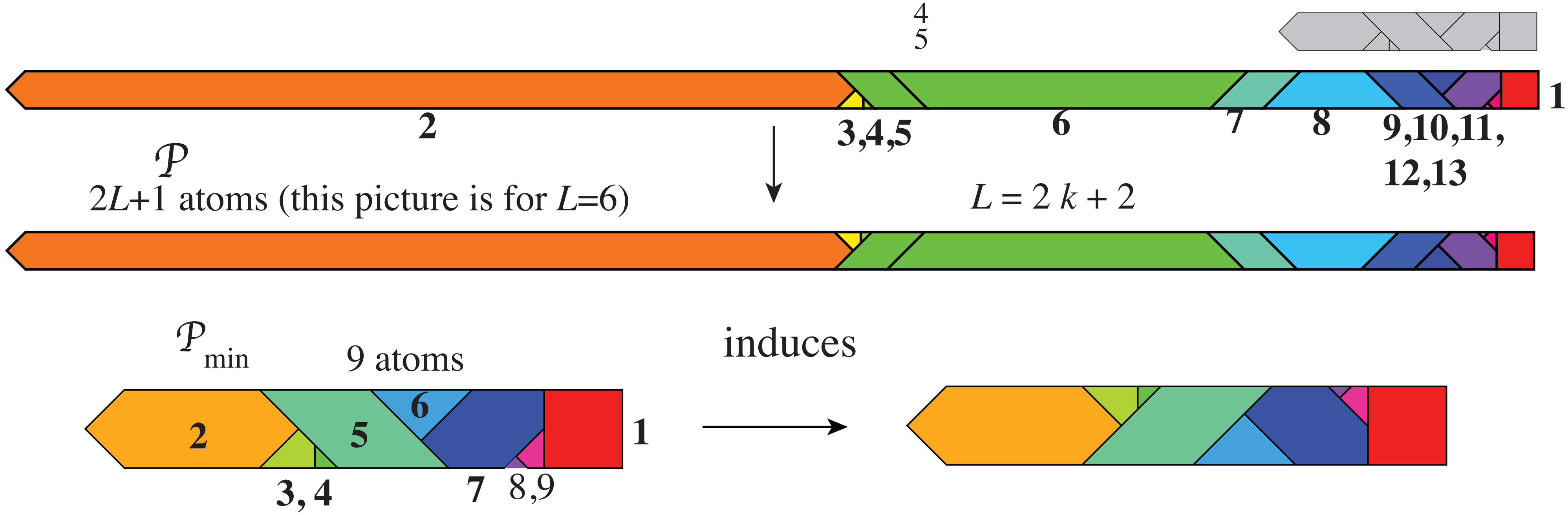
$a_k = -1/2 + (-1)^k + 1/2\ 3^{k-2}$

$b_k = 1 - 2\ (-1)^k + 3^{k-2}$

$c_k = -1/2 + 1/4(-1)^k + 1/4\ 3^{k-2}$

$d_k = 1 - 1/2\ (-1)^k + 1/2\ 3^{k-2}$

Step 3: $\mathcal{P}$ induces $\mathcal{P}_{\min}$



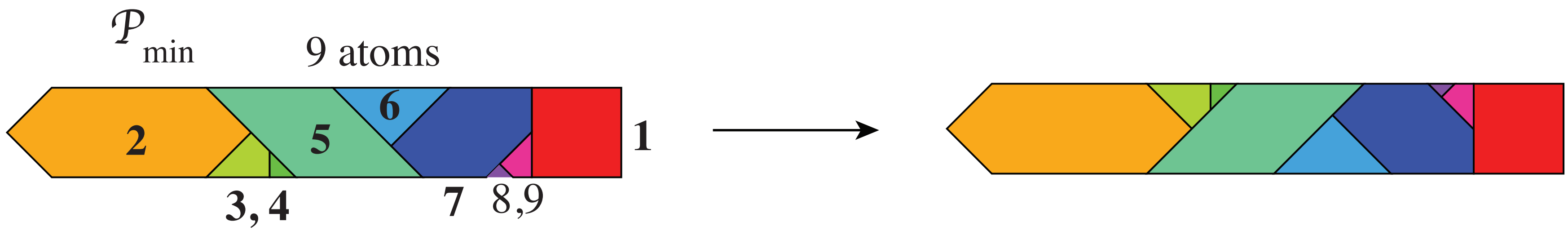
Return paths

1: 1
2: 2 L - 4
3: 2L-4, 2L-5, . . . 6, 5, 3, 5, 6, . . . , 2L-4
4: 2L-4, 2L-5, 2L-5, 2L-4
5: 2L-3
6: 2L-2
7: 2L-1
8: 2L
9: 2L+1

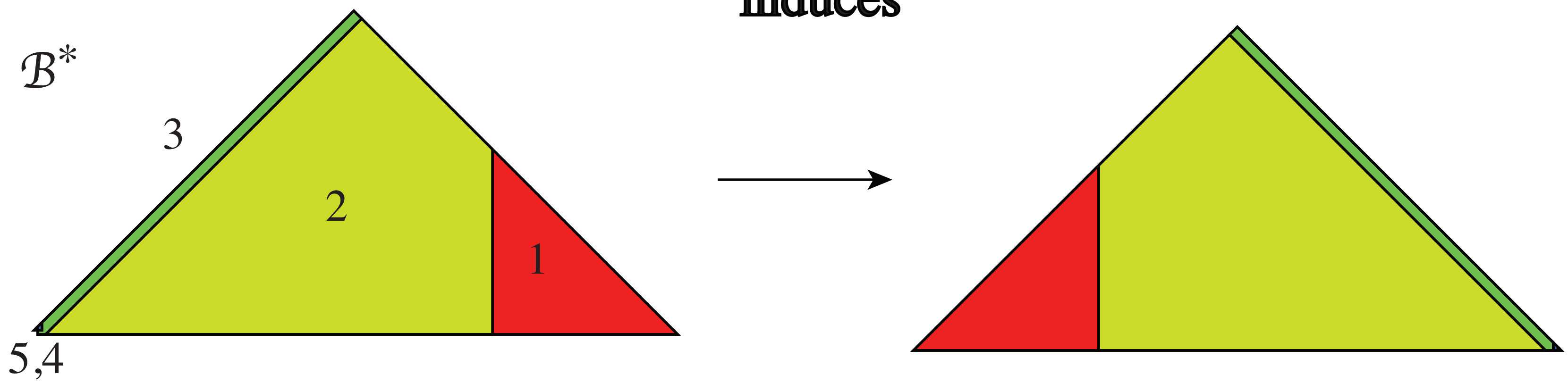
Incidence matrix

$$\left(\begin{array}{cccccccccc|ccccc}
1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & & & 0 & & \\
0 & 0 & 1 & 0 & 2 & 2 & \cdots & 2 & 2 & 2 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 & 2 & & & & & \\
\hline
 & & & & & & & & & & 1 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 1 & 0 & 0 & 0 \\
 & & & & & 0 & & & & & 0 & 0 & 1 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

And finally, step 4: $\mathcal{P}_{\min}$ induces $\mathcal{B}^*$



induces



Return paths

1: $8,7^2,9,7,6,(5,3,5,6)^6,7,9,7,7$

2: $8,7^2,9,7,6,(5,3,5,6)^3,7,9,7,7$

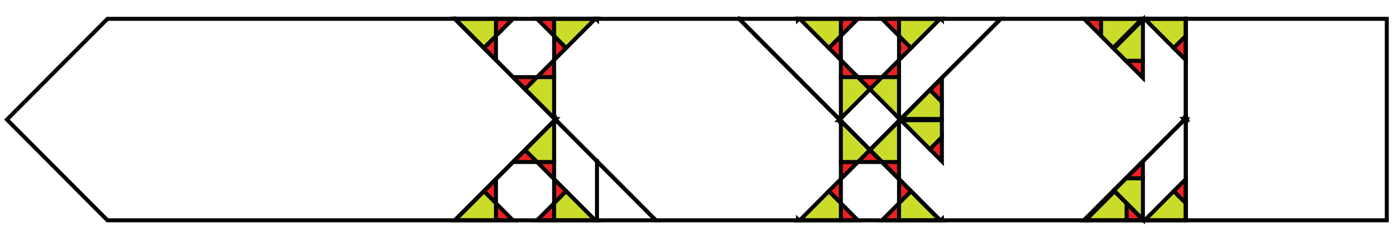
3: $8,7^2$

4: $8,7,6,5,3,5,6,7,9^3,7,6,5,3,5,6,7$

5: $8,7,6,5,3,5,6,7,9^6,7,6,5,3,5,6,7$

Incidence matrix

$$\begin{pmatrix} 0 & 0 & 6 & 0 & 12 & 7 & 6 & 1 & 2 \\ 0 & 0 & 3 & 0 & 6 & 4 & 6 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 & 4 & 4 & 4 & 1 & 3 \\ 0 & 0 & 2 & 0 & 4 & 4 & 4 & 1 & 6 \end{pmatrix}$$



Global Recursion Matrix, $\mathcal{B} \longrightarrow \mathcal{P}$

$$M_{\mathcal{B}\mathcal{P}} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1+i & 0 & -i & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & i & 0 & 0 \\ 0 & 1+i & -1 & 0 & 0 \\ 0 & i & -i & 0 & 0 \\ \multicolumn{5}{l}{\underline{n=k-1 \text{ odd:}}} \\ \frac{5-n}{2} & \frac{n-3}{2}(1+i) & -1 & 0 & 0 \\ \multicolumn{5}{l}{\underline{n=k-1 \text{ even, } n/2 \text{ odd}}} \\ \frac{6-n}{4}(1+i) & \frac{n-4}{2}\, i & -i & 0 & 0 \\ \multicolumn{5}{l}{\underline{n=k-1 \text{ even, } n/2 \text{ even}}} \\ \frac{n-4}{4}(i-1) & \frac{n-2}{2} & i & 0 & 0 \end{pmatrix}$$

Global Recursion Matrices, $\mathcal{P} \longrightarrow \mathcal{P}_{min}$, $\mathcal{P}_{min} \longrightarrow \mathcal{B}^*$

$$
M_{\mathcal{P}\mathcal{P}min} = \left(\begin{array}{ccccccccccccc|ccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & --- & 2L\text{-}7 & 2L\text{-}6 & 2L\text{-}5 & 2L\text{-}4 & 2L\text{-}3 & 2L\text{-}2 & 2L\text{-}1 & 2L & 2L+1 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & --- & 0 & 0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & --- & 0 & 0 & 0 & 1 & & & \mathbf{0} & & \\
0 & 0 & 1 & 0 & 21 & 1+i & 2 & 1+i & --- & 2i & 1+i & 2 & 1+i & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & --- & 0 & 0 & 1+i & 0 & & & & & \\
\hline
 & & & & & & & & & & & & & 1 & & & & \\
 & & & & & & & & & & & & & & 1 & & & \\
 & & & & & & \mathbf{0} & & & & & & & & & 1 & & \\
 & & & & & & & & & & & & & & & & 1 & \\
 & & & & & & & & & & & & & & & & & 1
\end{array}\right)
$$

$$
M_{\mathcal{P}min\mathcal{B}^*} = \begin{pmatrix}
0 & 0 & 1+i & 0 & -2+2i & i & 3+31 & -1 & -1+i \\
0 & 0 & 1 & 0 & 2i & 0 & 2 & i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1+i & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 1 \\
0 & 0 & 1-i & 0 & 2+2i & 2 & 2-2i & 1 & 1+i
\end{pmatrix}
$$

Cumulative Global Recursion Matrix for $\mathcal{B} \longrightarrow \mathcal{B}^*$

$$M_{\mathcal{B}\mathcal{B}^*} = M_{\mathcal{P}\min\mathcal{B}^*} M_{\mathcal{P}\mathcal{P}\min} M_{\mathcal{B}\mathcal{P}} = M_0 + M_1 L + M_2 L^2$$

$$M_0 = \begin{pmatrix} -3-16\,i & -7+11\,i & 2+8\,i & -1-i & 0 \\ 10-13i & 2+14\,i & 6+4i & -1 & 0 \\ -3+4i & 5-i & 2-i & 0 & 0 \\ 1-4i & -3+2i & 2i, & 0 & 0 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 1+12i & 7-9i & -4i & 0 & 0 \\ \frac{13+15i}{2} & -1-10i & -2-2i & 0 & 0 \\ 1-i & -2 & 0 & 0 & 0 \\ \frac{-1+3i}{2} & 2-i & 0 & 0 & 0 \\ 11 & -7-7i & -4 & 0 & 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} -2i & -2+2i & 0 & 0 & 0 \\ -1-i & 2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 2+2i & 0 & 0 & 0 \end{pmatrix}$$

Global expansion factor:

$$\gamma(L) = \left|\text{ largest eigenvalue of } M_{\mathcal{BB}*}\right|$$

$$= \left|\text{ largest root of } x^3 + a\,x^2 + b\,x + c\right|$$

$a = 3 - 2L$
$b = -19+18\,L-4L^2$
$c = 3$

$$= \left(-a+(a^2-3b)\,d^{-1/3} + d^{1/3}\right)/3$$

$$2\,d = -2\,a^3 + 9ab - 27c + 3\sqrt{3(-a^2b^2+4\,b^3+4\,a^3\,c - 18\,abc + 27\,c^2)}$$

Asymptotic large-time power = Log $\gamma(L)$/ Log $\eta(L)$,
where $\eta(L)$ is the temporal scaling factor, i.e. the largest eigenvalue of the cumulative incidence matrix $A_{\mathcal{BB}*}$

$$\eta(L) = \text{largest root of } x^3 + a\,x^2 + b\,x + c$$

$$= \left(-a+(a^2-3b)\,d^{-1/3} + d^{1/3}\right)/3$$

where

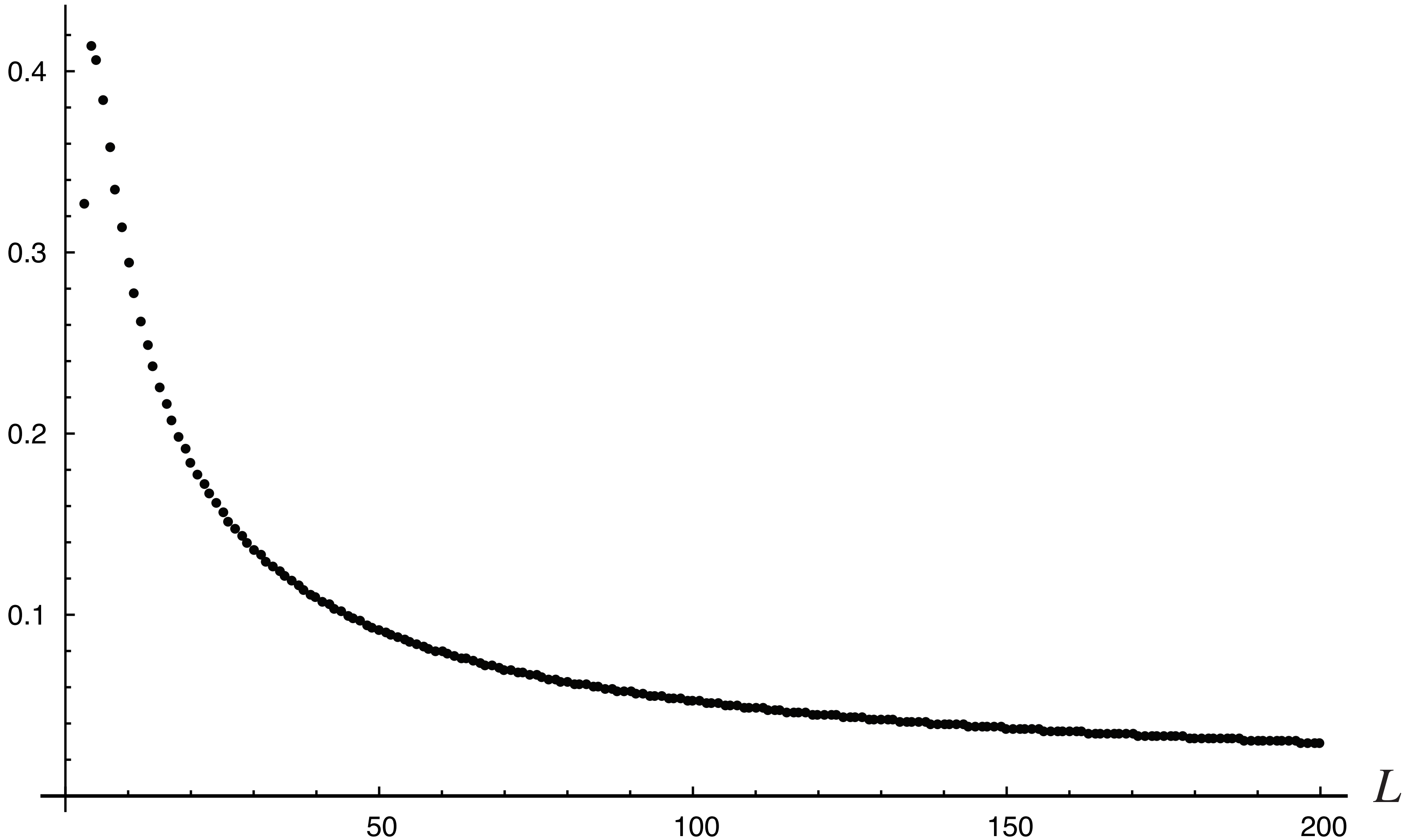
$a = -1-10 \times 3^{L-2}$

$b = -56 + 253 \times 3^{L-2} - 72 \times 3^{L-2}\,L$,

$c = 29 \times 3^{L-2}$

Note: as L tends to infinity, the global expansion factor increases only quadratically, so that the asymptotic power tends to zero.

Asymptotic powers for long-time behavior



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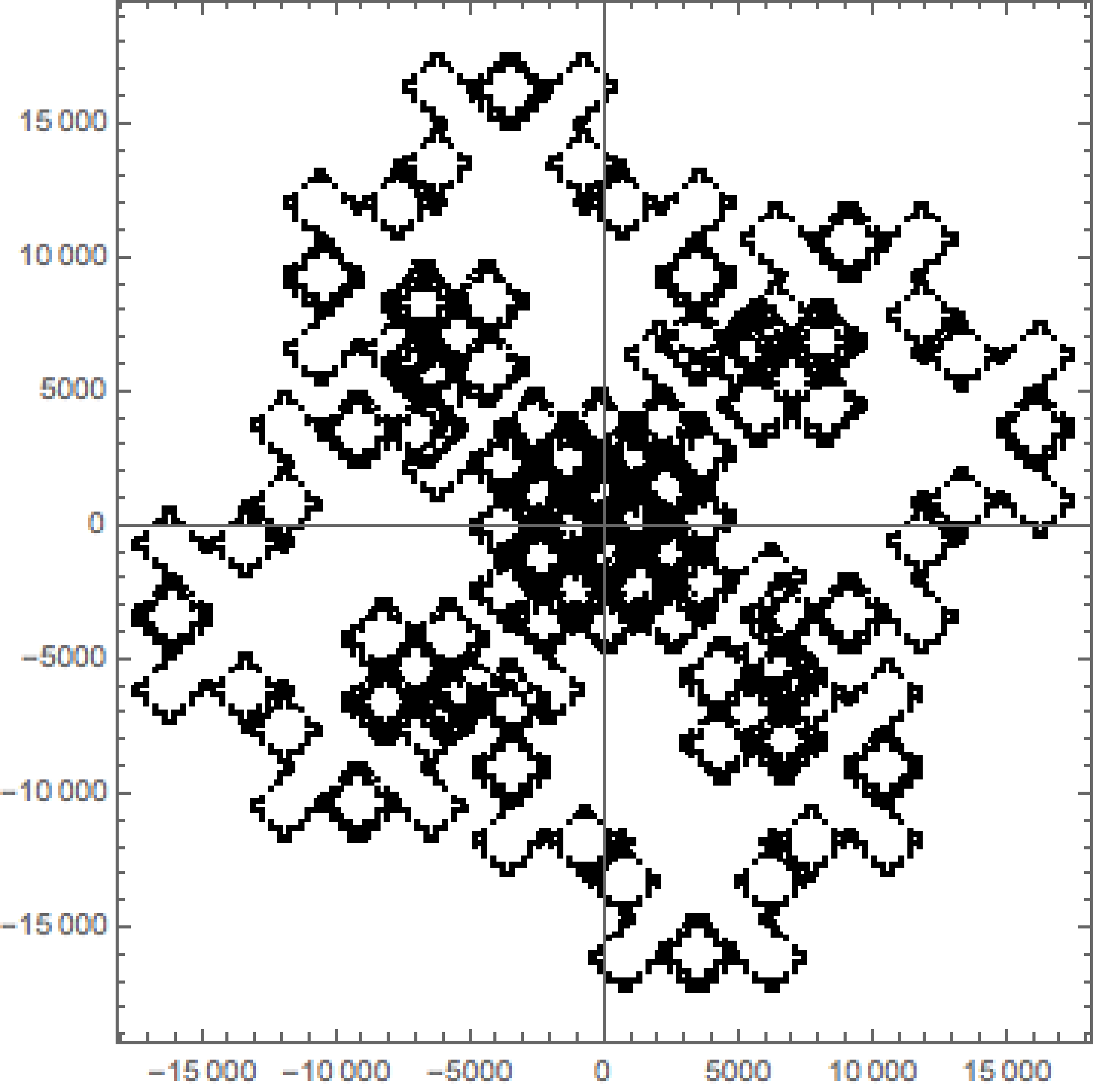
   2, 1, 2, 2, 2, 2, 1, 2, 1, 2, 2}};
```

```
update := Module[{k, m},
  For[k = 1, k ≤ Length[codometer], k++,
   If[codometer[[k, 3]] == ν[codometer[[k, 1]]][[codometer[[k, 2]]]] - 1,
    If[k == Length[codometer], Return["Overflow"], Continue[]]];
   codometer[[k, 3]]++;

   For[m = k - 1, m ≥ 1, m--,
    codometer[[m, 2]] = p @@ codometer[[m + 1]];
    codometer[[m, 3]] = 0]; Break[]]]


d1 = {2 - 20 ⅈ, -8 - 10 ⅈ, 2 - 3 ⅈ, 1 + ⅈ, -19};

codometer = {{1, 3, 0}, {0, 3, 0}, {1, 3, 0}, {0, 3, 0}, {1, 3, 0}};
orbit = Table[0, {100 000}];
step = 1;
z = 0;
Do[update; step++; j = p @@ (codometer[[1]]);
  z = I ^ (τ[0][[j]]) z + d1[[j]];
 orbit[[step]] = z , {99 999}];
ListPlot[{Im[#], Re[#]} & /@ orbit, PlotStyle → {Black, PointSize[.005]},
 AspectRatio → Automatic, Frame → True]
```


15 000
10 000
5000
0
−5000
−10 000
−15 000
−15 000
−10 000
−5000
0
5000
10 000
15 000