Chaire Jean-Morlet -- Akiyama-Arnoux Semester--Small Group Conference on Zero-Entropy Systems
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Pseudochaotic Kicked-Oscillator Maps and Parametric Piecewise Isometries

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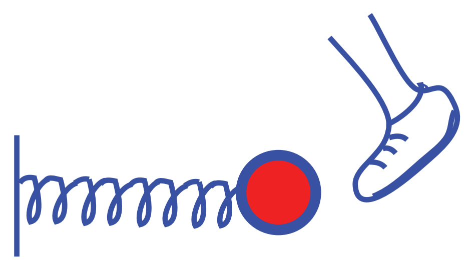
Starting point:



Principal references: J.H.Lowenstein and F. Vivaldi,
Dyn. Sys. **31**, 393-465 (2016), *Chaos,* **26**, 663119 (2016)

Hamiltonian and Equations of Motion

$$H(q,p) = \tfrac{1}{2}(p^2 + q^2) - F(q) \sum_n \delta(t - 2\pi\rho n)$$

ρ = rotation number = 1/# kicks per natural period
(Resonant case: ρ rational)

$$F(q) = F(q + \tau)$$

$$q = \frac{\partial H}{\partial p} = p$$

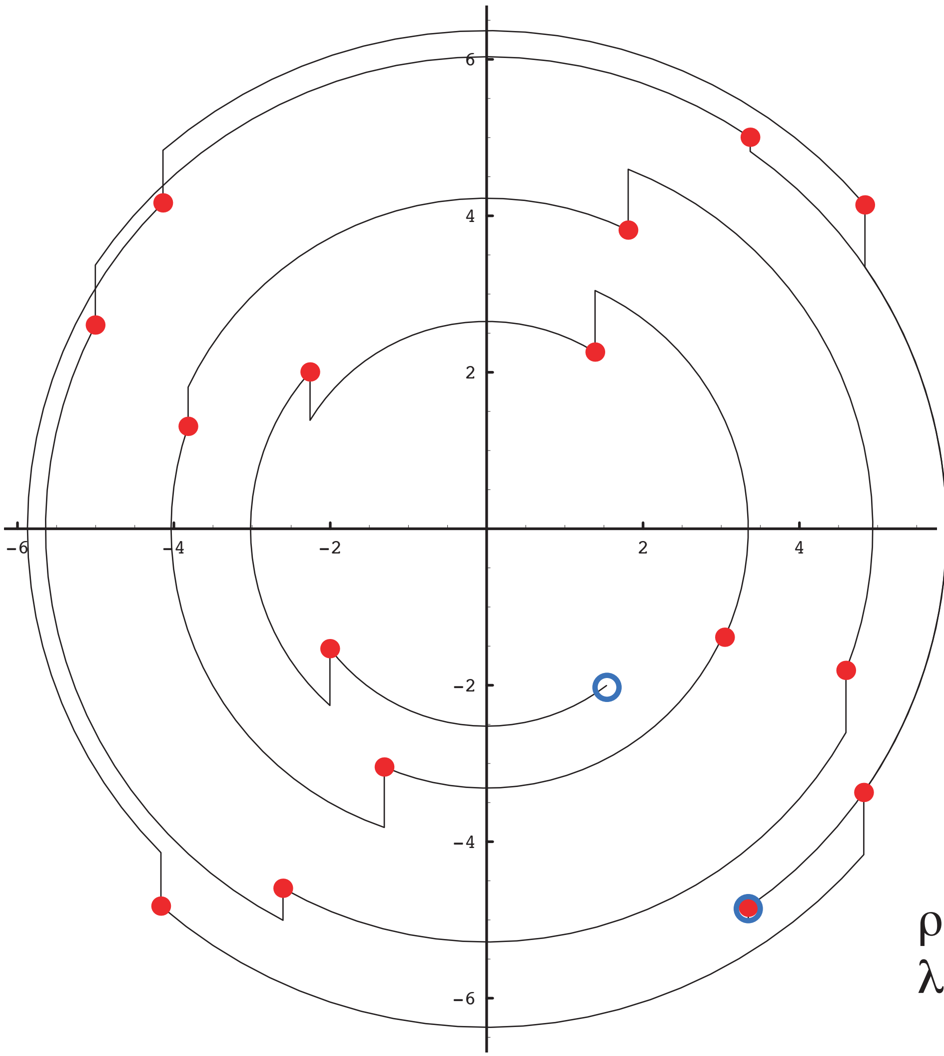
$$p = -\frac{\partial H}{\partial q} = -q + f(q) \sum_n \delta(t - 2\pi\rho n)$$

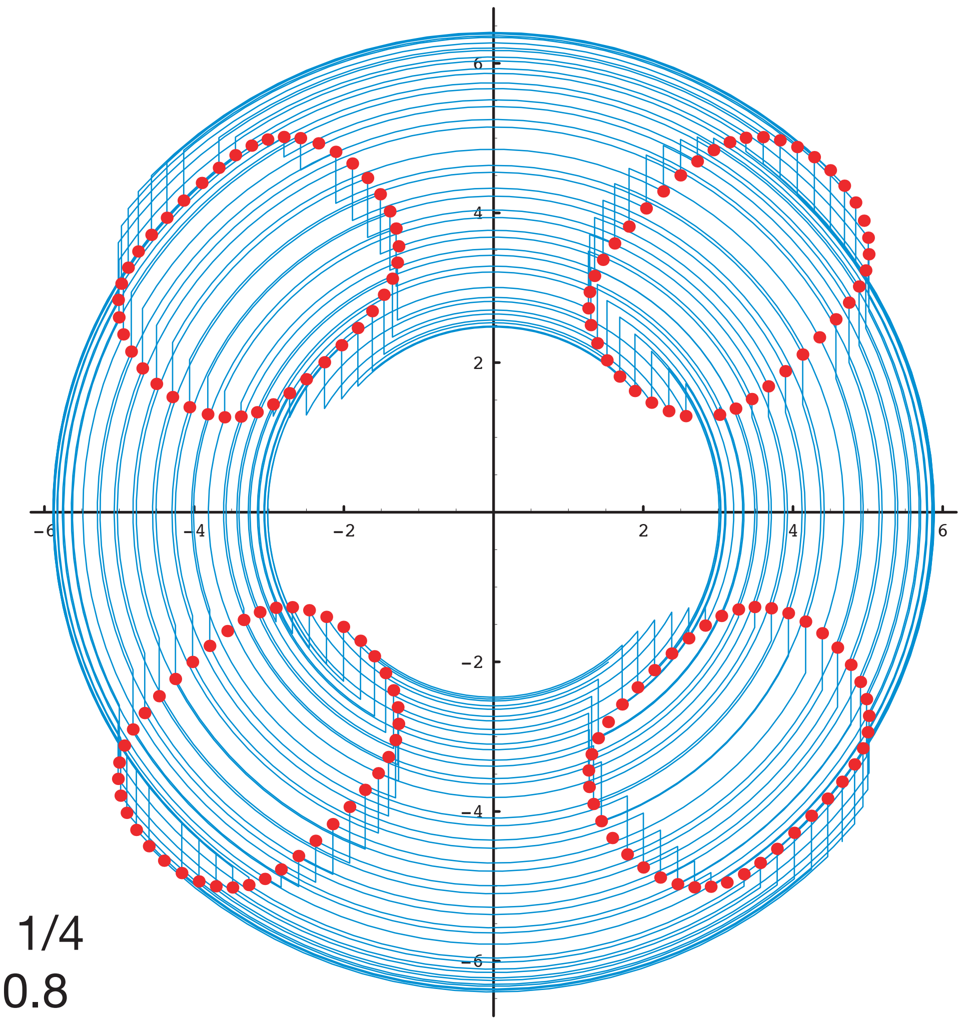
Free oscillation for fraction ρ of a natural period, followed by momentum shift $p \dashrightarrow p + \Delta p$, $\Delta p = f(q) = F'(q)$ = "kick amplitude"

Initial choice: $F(q) = \lambda \cos(q)$, $\Delta p = \lambda \sin(q)$, $\tau = 2\pi$

Example: $\lambda = 0.8$, quasi-periodic orbit

Quasi-periodic Phase-Space Orbit and Poincaré Section





$\rho = 1/4$

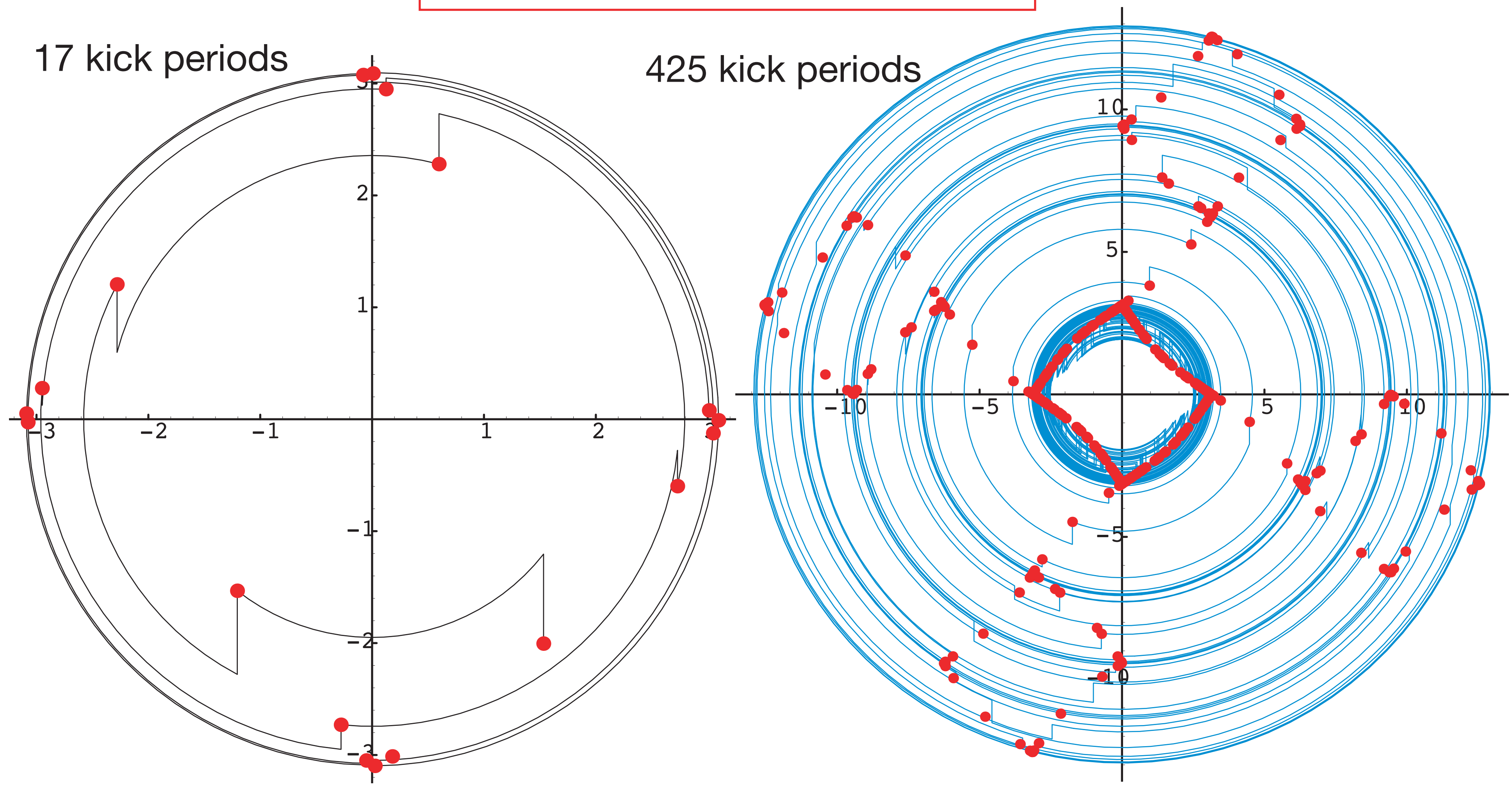
$\lambda = 0.8$

Stroboscopic view in phase space: plot (q,p) just before each kick.

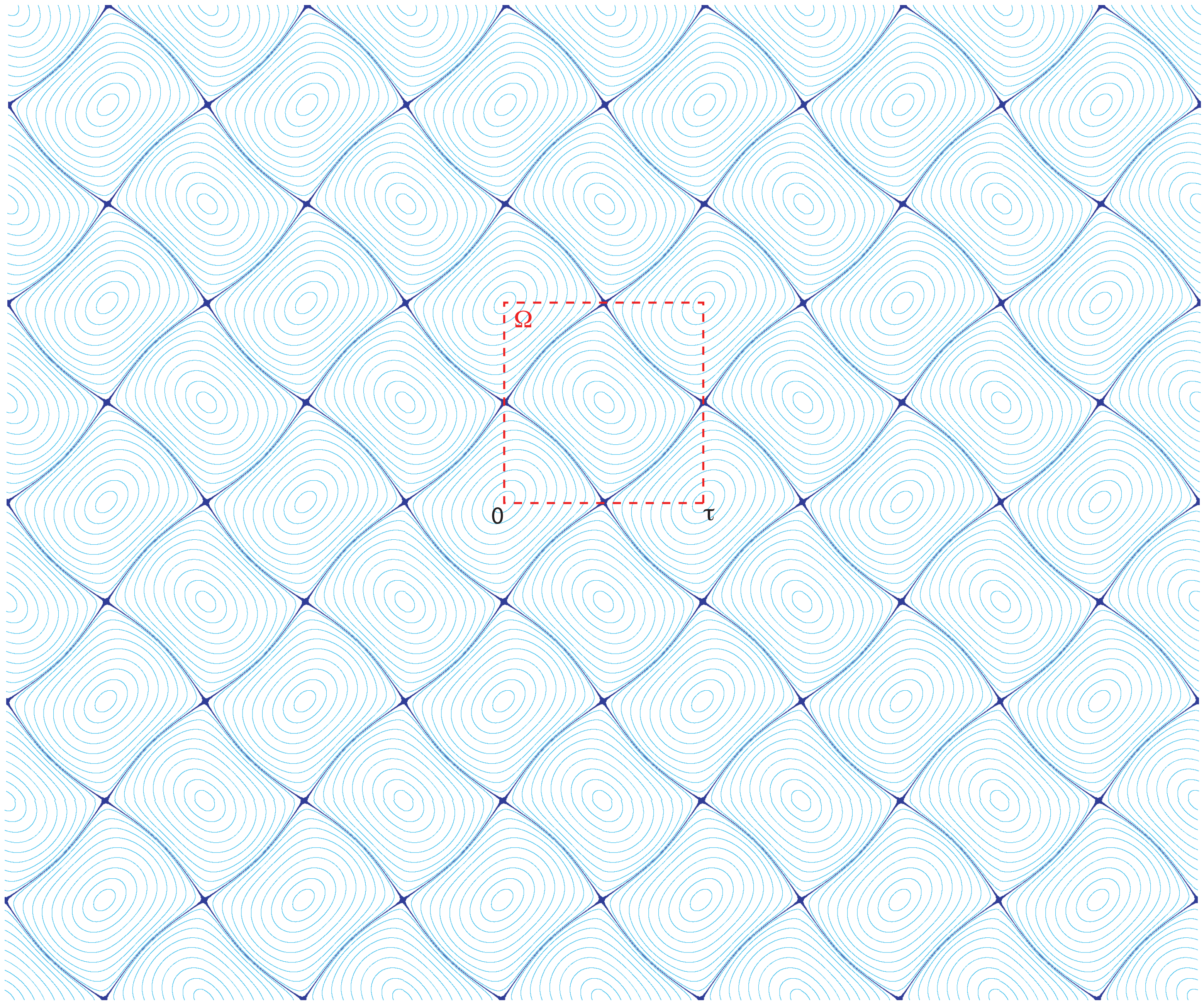
Poincaré map

$$\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \cos 2\pi\rho & \sin 2\pi\rho \\ -\sin 2\pi\rho & \cos 2\pi\rho \end{pmatrix} \begin{pmatrix} q \\ p + \lambda \sin q \end{pmatrix} = \begin{pmatrix} p + \lambda \sin q \\ -q \end{pmatrix}$$

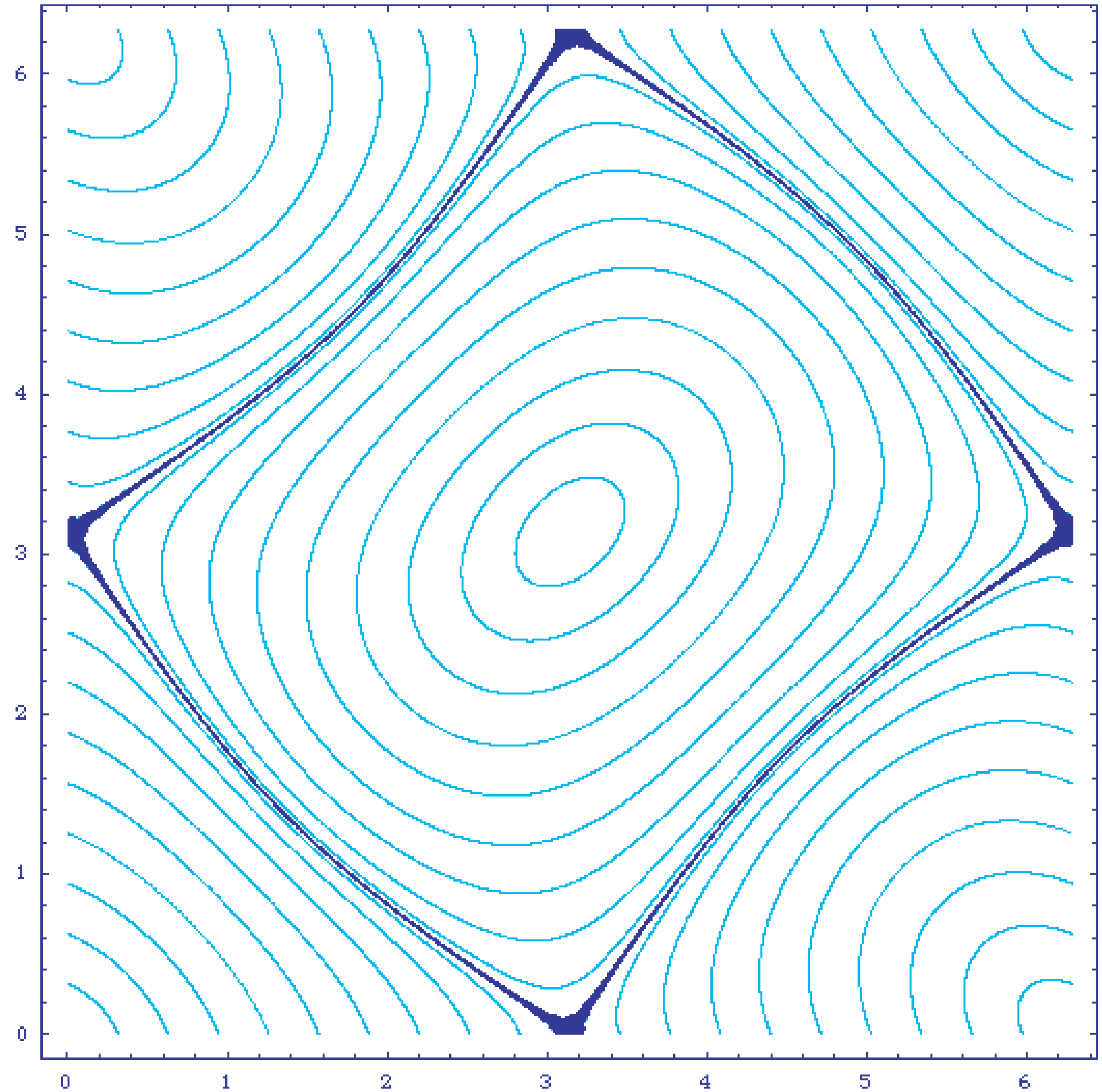
Chaotic Orbit and Poincaré Section



Stochastic Web with 4-fold Crystalline Symmetry



Phase Portrait of Local Stochastic Web Map, λ=0.8



Normal and Anomalous Diffusion

For typical values of λ , the chaotic orbits of W^4 in the stochastic web proceed to infinity with mean-square distance from the initial point satisfying

$$\langle q^2 + p^2 \rangle \sim D\, t \quad \text{for } t \longrightarrow \infty$$

D = diffusion constant

More general power law behavior:

$$\langle q^2 + p^2 \rangle \sim D'\, t^{\mu} \quad \text{for } t \longrightarrow \infty$$

diffusion: $\mu = 1$
super-diffusion: $\mu > 1$
sub-diffusion: $\mu < 1$

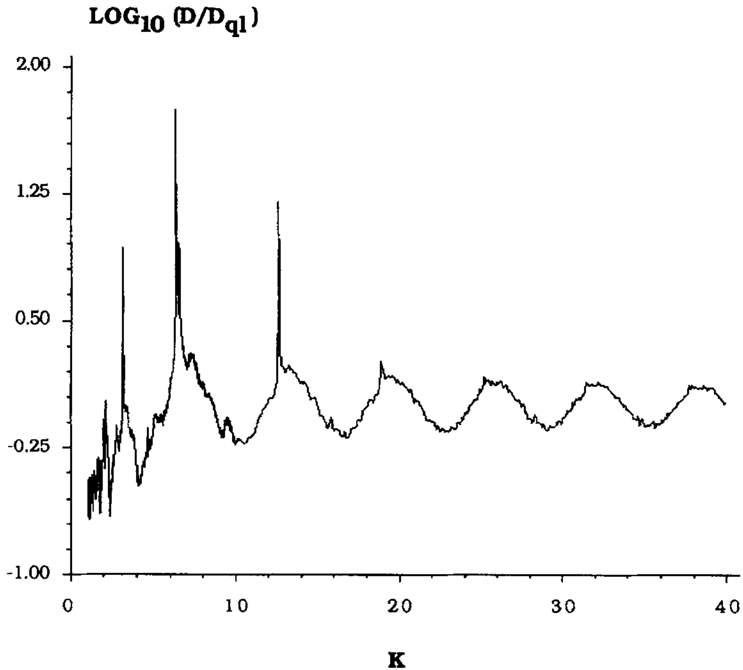
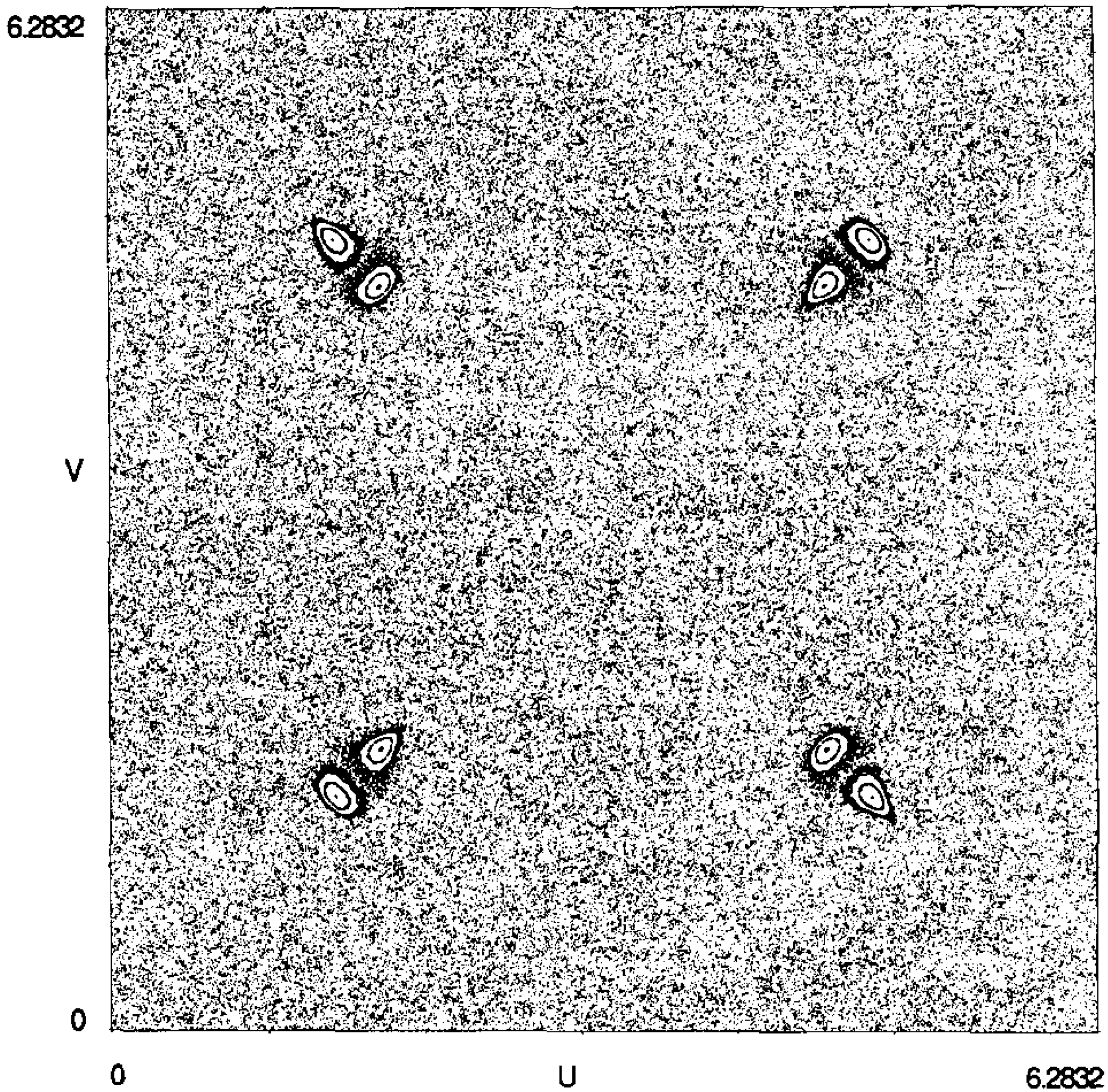
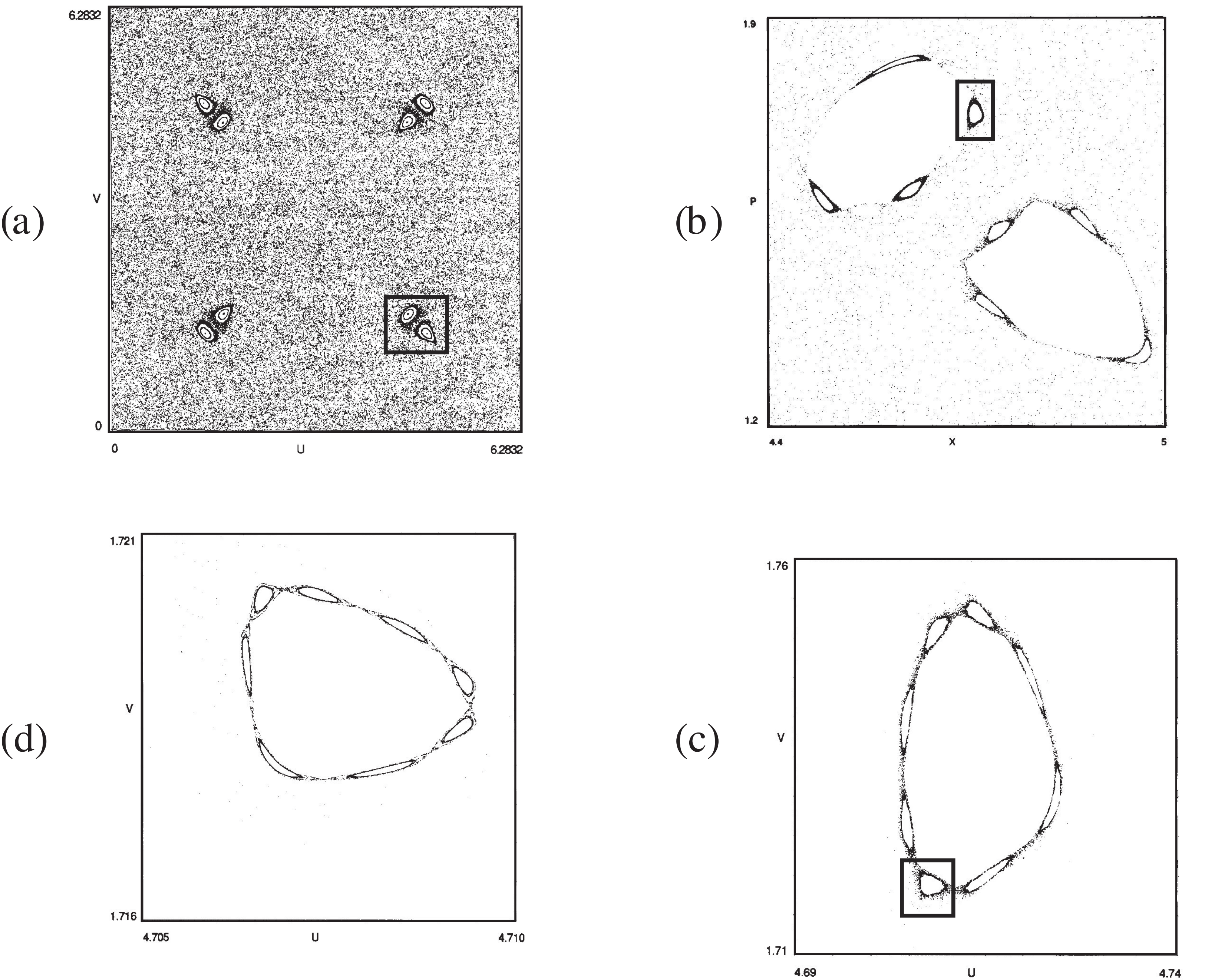


Fig. 2. Diffusion coefficient $\mathscr{D}$ normalized to the quasilinear theory coefficient $\mathscr{D}_{\rm ql} = K^2/2$ for different values of K, obtained after averaging over 2500 trajectories for each value of K with 0.5×10^5 iterations for each trajectory.

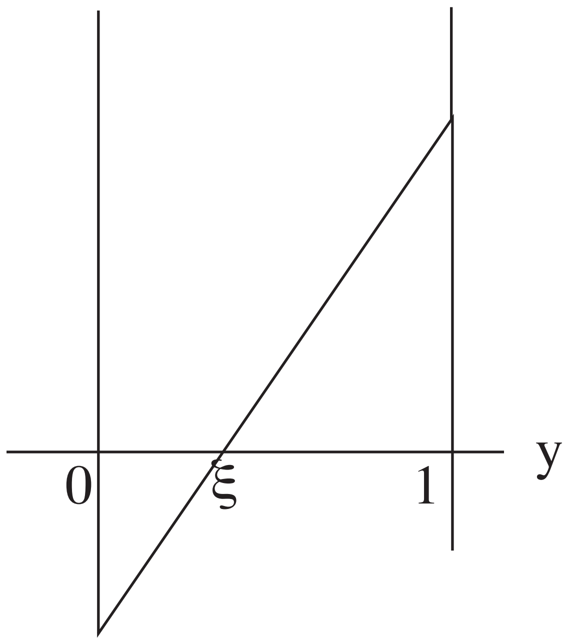

6.2832
V
0
0
U
6.2832

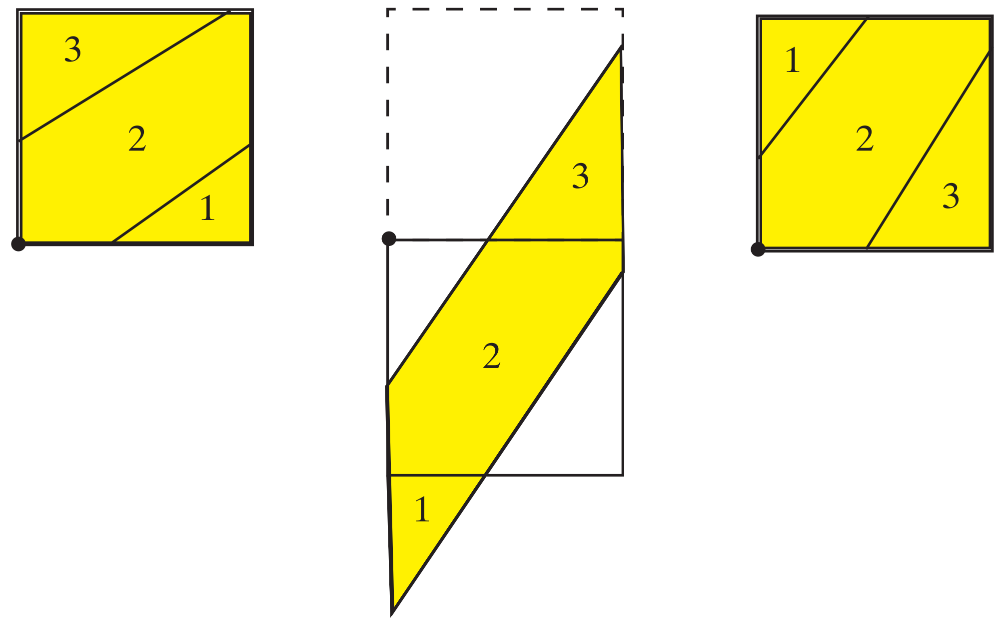

(a)
6.2832
V
0
0
U
6.2832
(b)
1.9
P
1.2
4.4
X
5
(d)
1.721
V
1.716
4.705
U
4.710
(c)
1.76
V
1.71
4.69
U
4.74

Global Kicked-Oscillator Map

$$W\begin{pmatrix} x \\ y \end{pmatrix} = R(-\pi/2)\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ f(y) \end{pmatrix} = \begin{pmatrix} y \\ -x + f(y) \end{pmatrix}$$

$$f(y) = \alpha\,(\,y \bmod 1 - \xi\,)$$



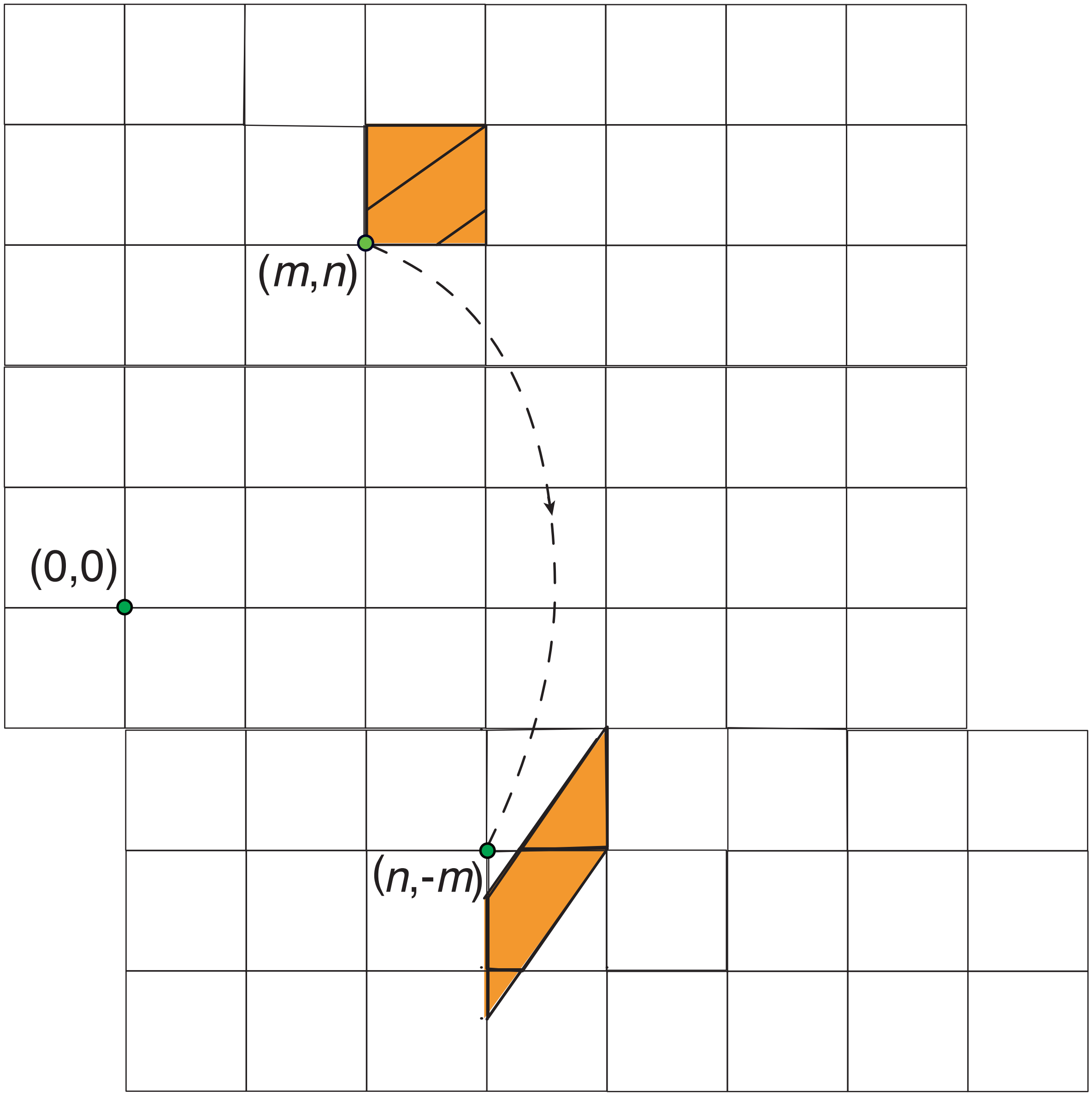


Local Map K

$$K\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ (-x + \alpha\, y - \xi) \bmod 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 3-\iota(x,y) \end{pmatrix}$$

piecewise affine transformation

Action of map W on the plane

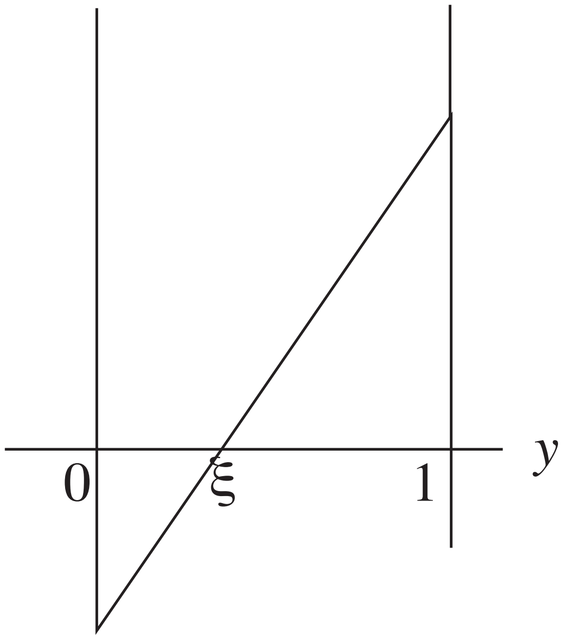


W decomposes into K plus a Z^2 lattice isometry

$$W(\boldsymbol{u} + \boldsymbol{m}) = K(\boldsymbol{u}) + \mathrm{R}(-\pi/2)\,\boldsymbol{m} + \mathrm{d}(\boldsymbol{u})$$

$$\mathrm{d}(\boldsymbol{u}) = \begin{cases} (0,-2) & \boldsymbol{u} \in \mathrm{D}_1 \\ (0,-1) & \boldsymbol{u} \in \mathrm{D}_2 \\ (0,0) & \boldsymbol{u} \in \mathrm{D}_3 \end{cases}$$

$$f(y) \;=\; \alpha\,(\,y \bmod 1 - \xi\,)$$



My choice of parameters:

slope α : $\sqrt{2}$

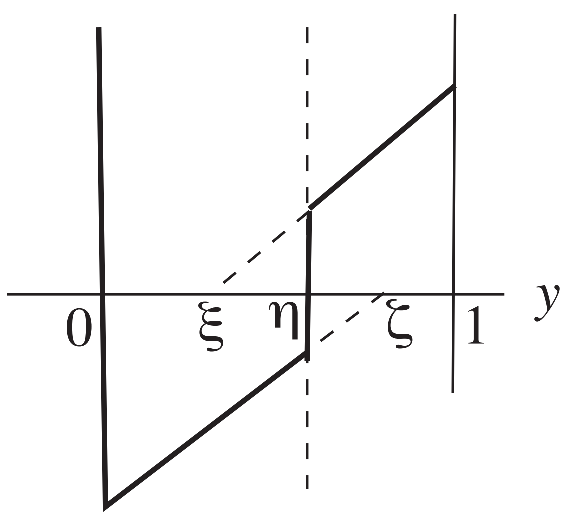
intercept $\xi = \dfrac{\beta}{2}(\alpha - \beta\, s\,),\quad 0 \leqslant s < \alpha$

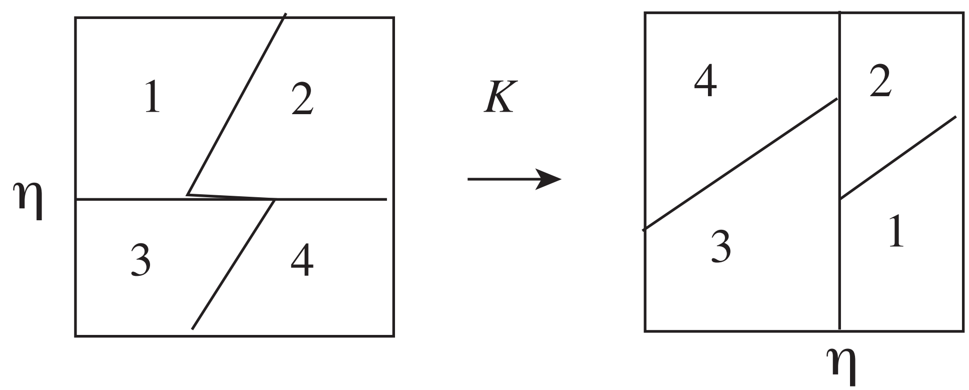
Notation:

$\beta = \alpha - 1$

$\omega = \alpha + 1$

More complicated $f(y)$:





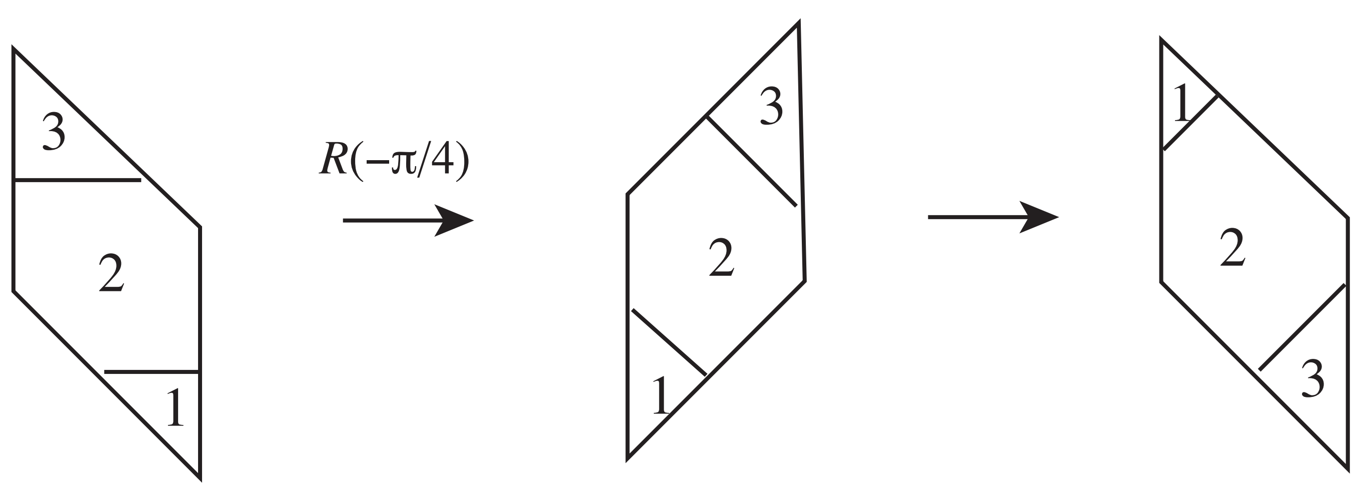
K expressed as a true piecewise isometry

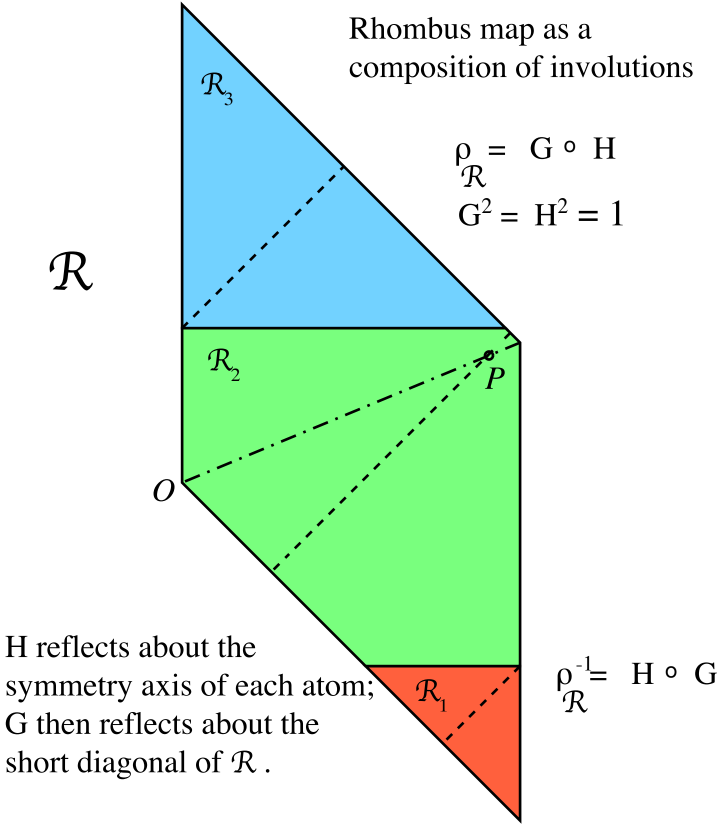
For $|\alpha| < 2$, $\begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix}$ has eigenvalues $e^{i\theta}$, $e^{-i\theta}$, hence is conjugate to

$$R(-\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

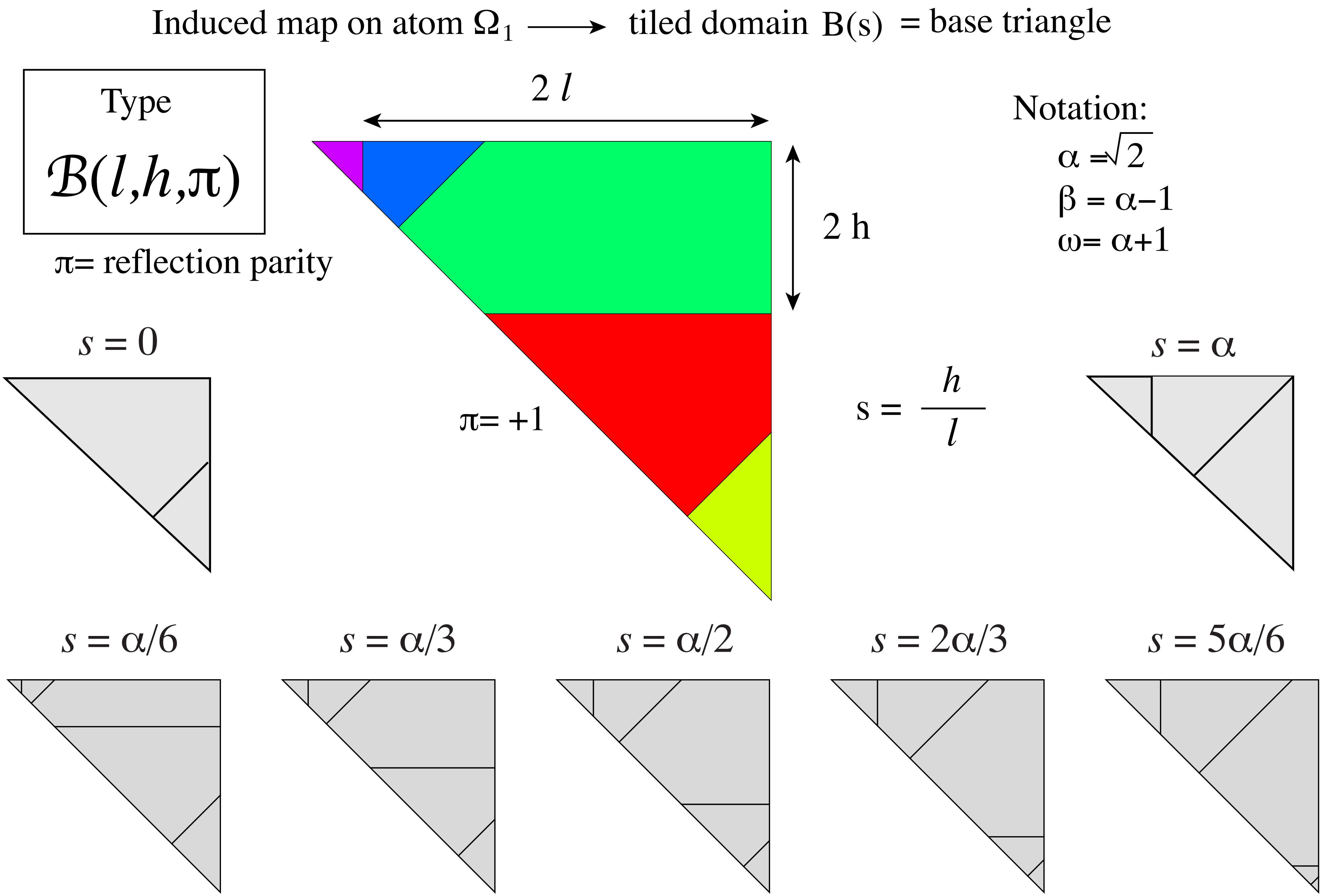
Hence K is conjugate to a rotation followed by a piecewise translation, applied to a rhombus with vertex angle θ .

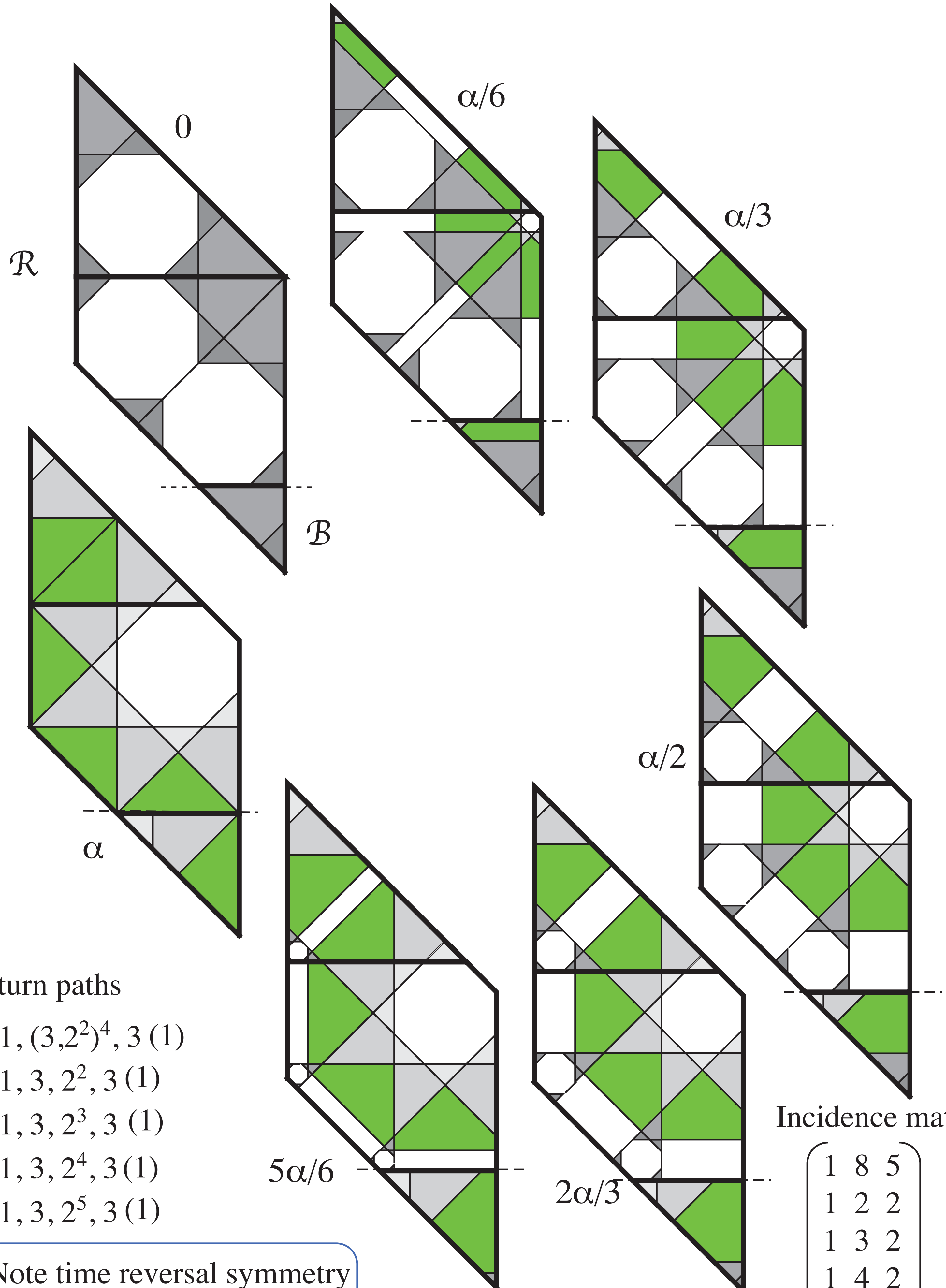
Example for $\alpha = \sqrt{2}$, $\theta = \pi/4$





Rhombus map as a composition of involutions
$\rho_{\mathcal{R}} = G \circ H$
$G^2 = H^2 = 1$
$\mathcal{R}$
$\mathcal{R}_3$
$\mathcal{R}_2$
P
O
$\mathcal{R}_1$
$\rho_{\mathcal{R}}^{-1} = H \circ G$
H reflects about the symmetry axis of each atom; G then reflects about the short diagonal of $\mathcal{R}$.


Induced map on atom Ω_1 ⟶ tiled domain B(s) = base triangle
Type
$\mathcal{B}(l,h,\pi)$
π= reflection parity
$2\,l$
$2\,h$
Notation:
$\alpha = \sqrt{2}$
$\beta = \alpha - 1$
$\omega = \alpha + 1$
π= +1
$s = \frac{h}{l}$
$s = 0$
$s = \alpha$
$s = \alpha/6$
$s = \alpha/3$
$s = \alpha/2$
$s = 2\alpha/3$
$s = 5\alpha/6$



Return paths

1: 1, $(3,2^2)^4$, 3 (1)

2: 1, 3, 2^2, 3 (1)

3: 1, 3, 2^3, 3 (1)

4: 1, 3, 2^4, 3 (1)

5: 1, 3, 2^5, 3 (1)

Note time reversal symmetry (palindromes!)

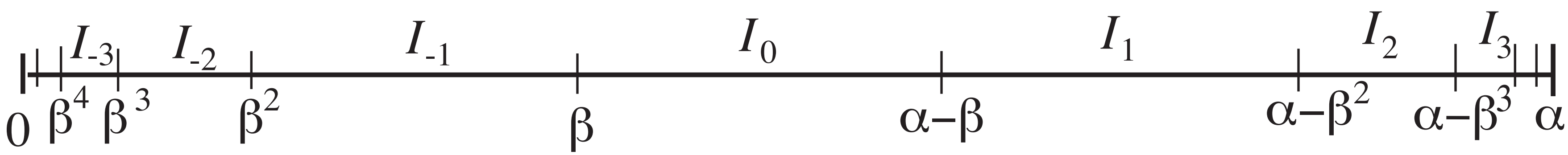
Incidence matrix

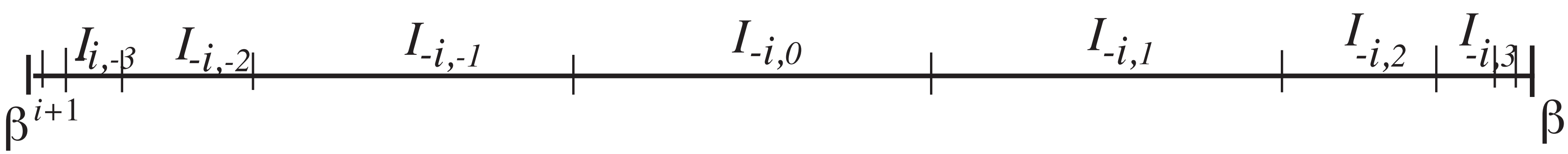
$$\begin{pmatrix} 1 & 8 & 5 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 4 & 2 \\ 1 & 5 & 2 \end{pmatrix}$$

Our goal :

Given s in $[0, \alpha]$, to find within the base triangle $\mathcal{B}(s)$ a higher-level base triangle $\mathcal{B}^*(s)$ conjugate to $\mathcal{B}(r(s))$ via a similarity transformation, for some suitable renormalization function r(s).

Numerical experiment gave strong evidence that this is true, with the following specification of r(s) (notation: $\alpha = \sqrt{2}$, $\beta = \alpha - 1$, $\omega = \alpha + 1$)



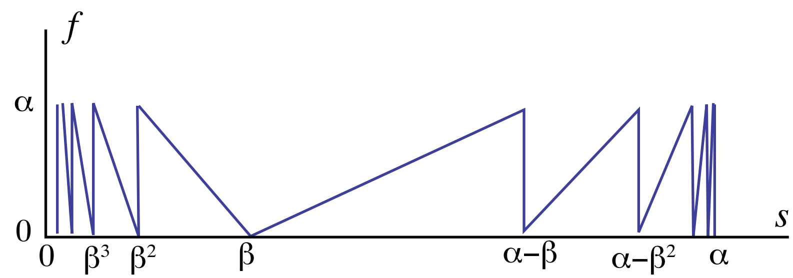


$$I_{-i,-j} = (\beta^{i+1} + \beta^{i+j+2}, \beta^{i+1} + \beta^{i+j+1}] \qquad i > 0,\ j \geq 0$$

$$I_{-i,j} = [\beta^{i} - \beta^{i+j+1}, \beta^{i} - \beta^{i+j+2}) \qquad i > 0,\ j > 0$$

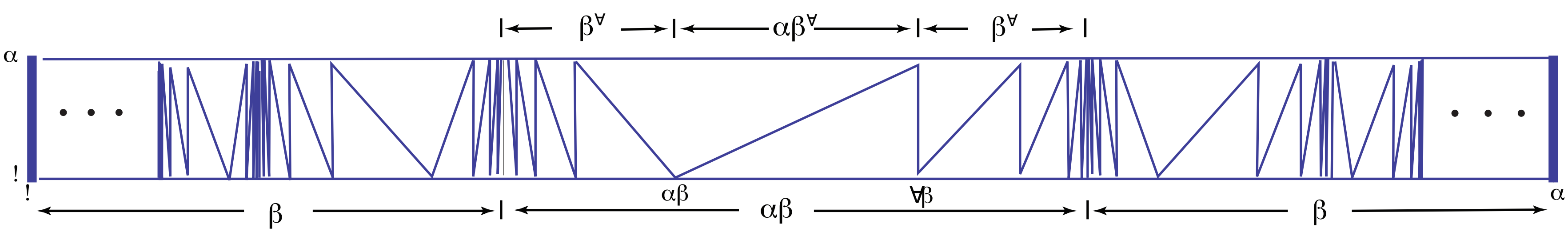
$$I_{0,0} = (\beta + \beta^2, 2\beta)$$

On I_i , $f(s) = \pm\,\omega^{|i|+1} \qquad {}_{i}\,)$



$$r(s) = f^{\,2}(s)$$

On I_{ij} , $r(s) = \pm\,\omega^{|i|+|j|+2}\ (\,s - \Delta_{ij}\,)$



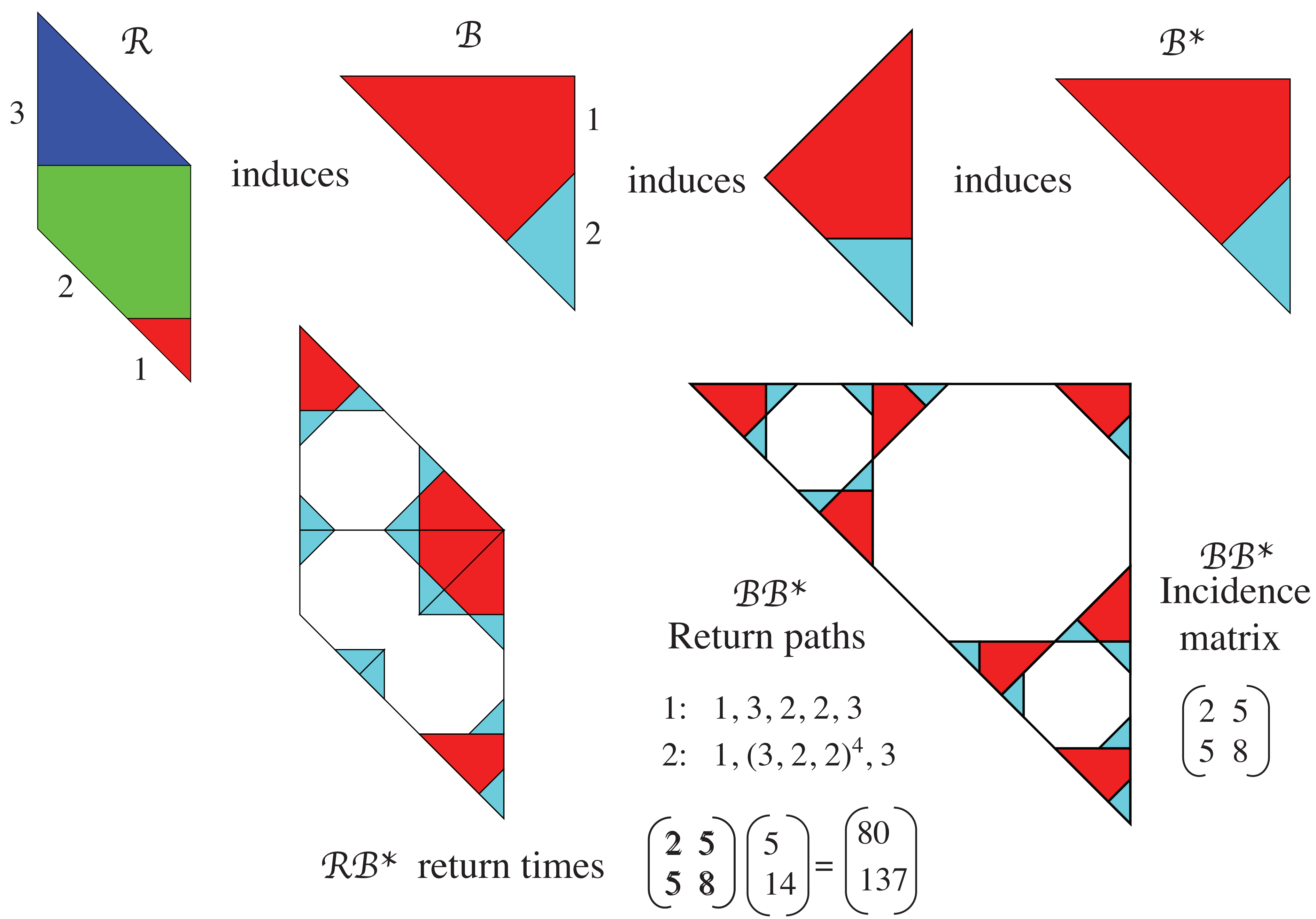
Main Results

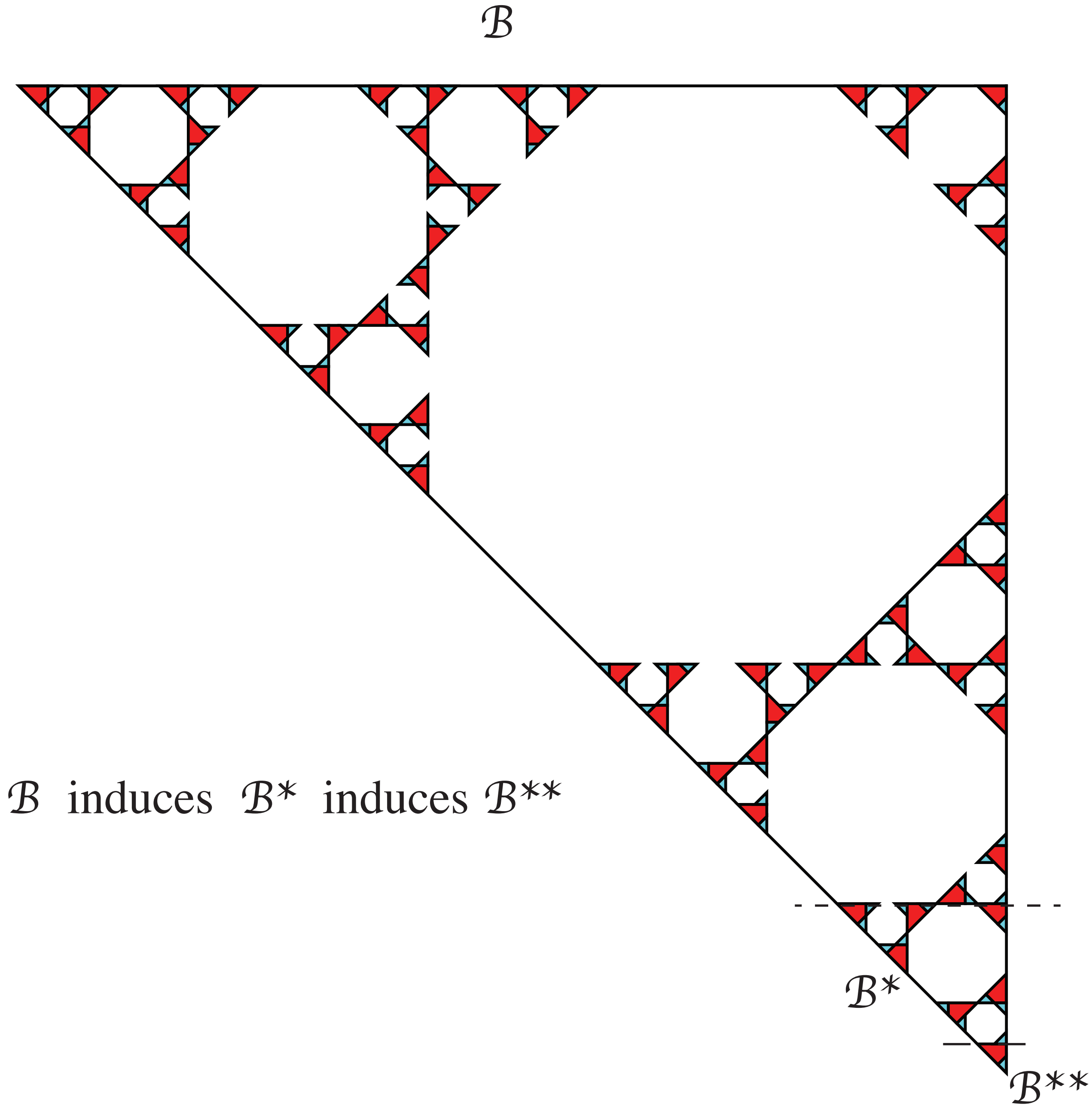
Theorem 1   Let $f(s)$ and $r(s) = f^2(s)$ be defined as above. Let $\mathcal{B}(s)$ be the base triangle with its 5-atom domain map. Then for all $s \in [0,\alpha)$, there exists a tiled domain $\mathcal{B}^*(s) \subset \mathcal{B}(s)$ such that $\mathcal{B}^*(s)$ is conjugate, via a similarity transformation, to $\mathcal{B}(r(s))$. Furthermore, with the exception of the accumulation points of $r(s)$, the base triangle is tiled by the return orbits of the atoms of $\mathcal{B}^*(s)$, up to a finite number of periodic domains.

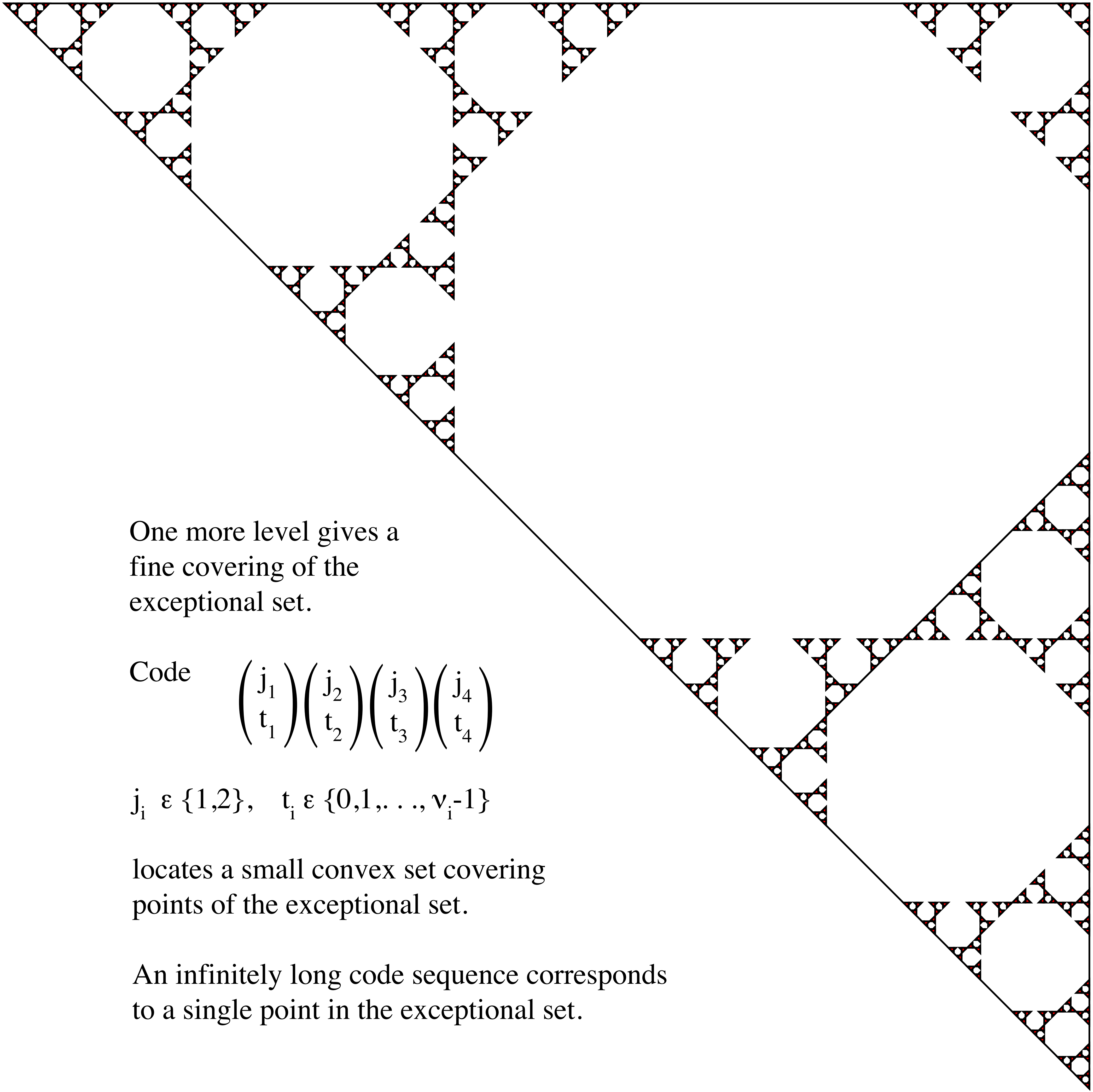
Theorem 2   A point $s \in I = [0,\sqrt{2})$ is eventually periodic under f if and only if $s \in Q(\sqrt{2})$. Hence the set of parameter values for which the dynamical map L is dynamically self-similar is $Q(\sqrt{2}) \cap I$.

Coding: $s \longleftrightarrow i_0\, i_1\, i_2 \ldots\ldots$ if $f^k(s) \in I_k$ $k=0,1,2,\ldots$

Renormalization fixed point $s = 0 = r(0)$




$\mathcal{B}$
$\mathcal{B}$ induces $\mathcal{B}^*$ induces $\mathcal{B}^{**}$
$\mathcal{B}^*$
$\mathcal{B}^{**}$



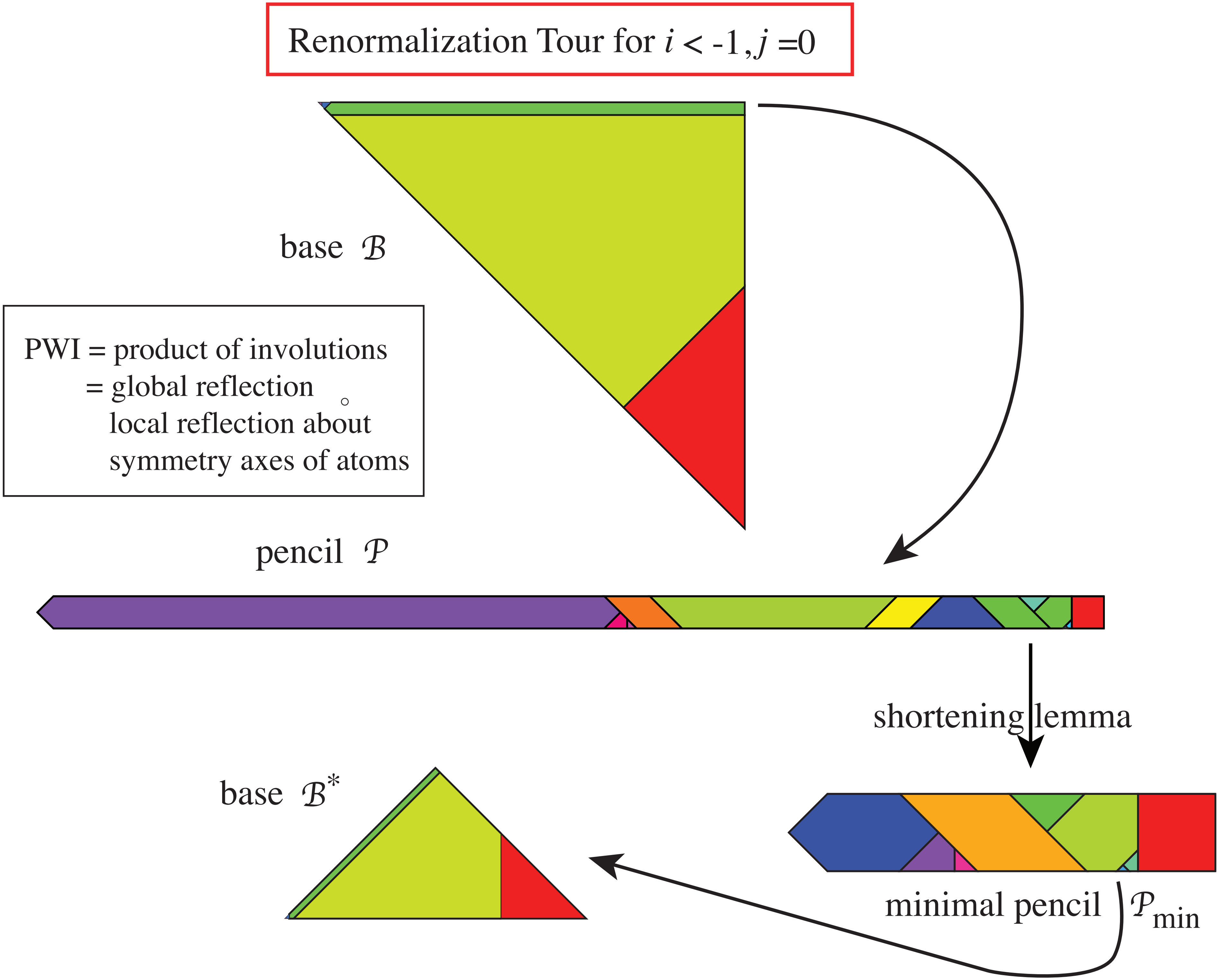
One more level gives a fine covering of the exceptional set.

Code $$\begin{pmatrix} j_1 \\ t_1 \end{pmatrix}\begin{pmatrix} j_2 \\ t_2 \end{pmatrix}\begin{pmatrix} j_3 \\ t_3 \end{pmatrix}\begin{pmatrix} j_4 \\ t_4 \end{pmatrix}$$

$j_i \; \varepsilon \; \{1,2\}, \quad t_i \; \varepsilon \; \{0,1,. . ., \nu_i\text{-}1\}$

locates a small convex set covering points of the exceptional set.

An infinitely long code sequence corresponds to a single point in the exceptional set.


Renormalization Tour for $i < -1, j = 0$
base $\mathcal{B}$
PWI = product of involutions
= global reflection $\circ$
local reflection about
symmetry axes of atoms
pencil $\mathcal{P}$
shortening lemma
base $\mathcal{B}^*$
minimal pencil $\mathcal{P}_{\min}$

Renormalization scenario $\mathcal{R} \to \mathcal{B} \to \mathcal{P} \to \mathcal{P}_{\min} \to \mathcal{B}^*$ on the parameter intervals

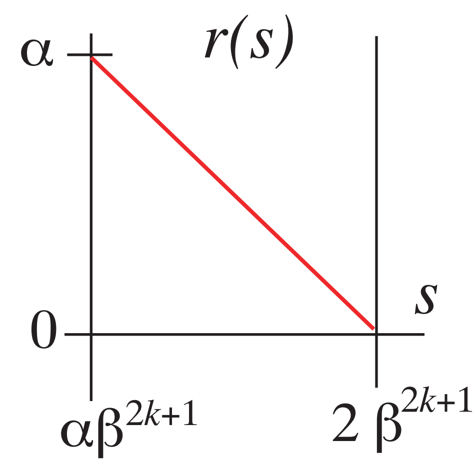
$$r^m(s) \;\varepsilon\; \mathrm{I}_{-2k,0} = (\alpha\, \beta_{2k+1}, 2\, \beta_{2k+1}), \qquad k = 1, 2, 3, \ldots, \; m = 0, 1, 2, \ldots$$

This is an r - invariant Cantor set whose Hausdorff dimension is determined by the equation

$$\sum_{k=0}^{\infty} \beta^{(2k+4)\,d} = \frac{\beta^{4\,d}}{1 - \beta^{2\,d}} = 1$$

$$d = \frac{\log \gamma}{2 \log \beta} \quad , \qquad \gamma = (\sqrt{5} - 1)/2$$

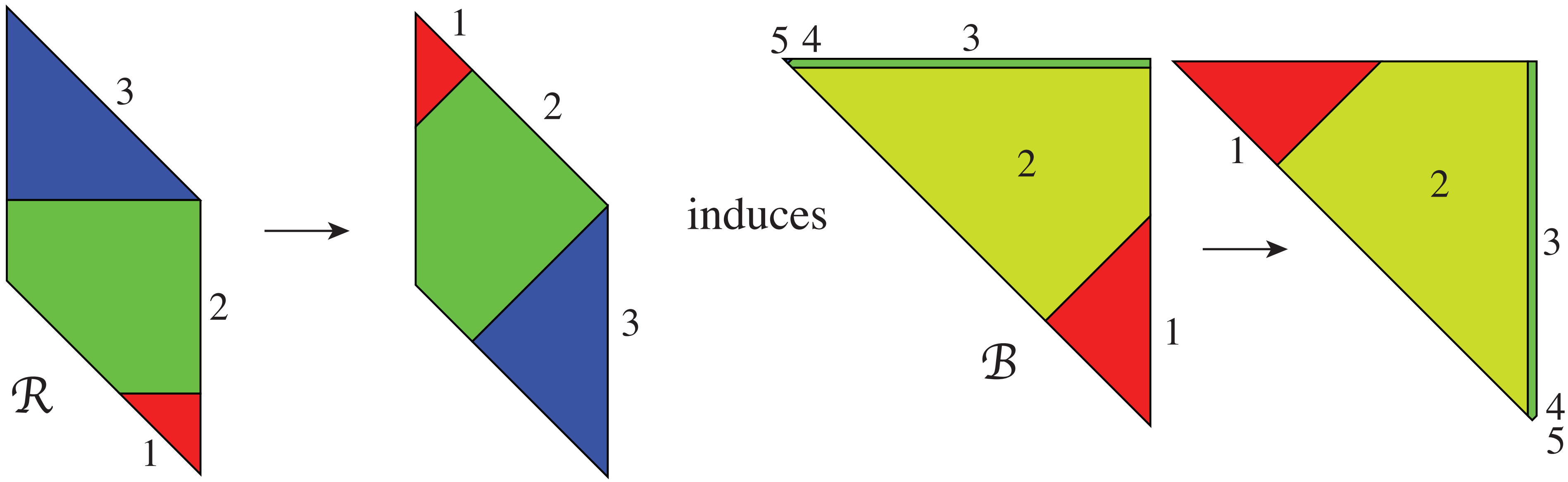
$$d = 0.2729897\ldots$$

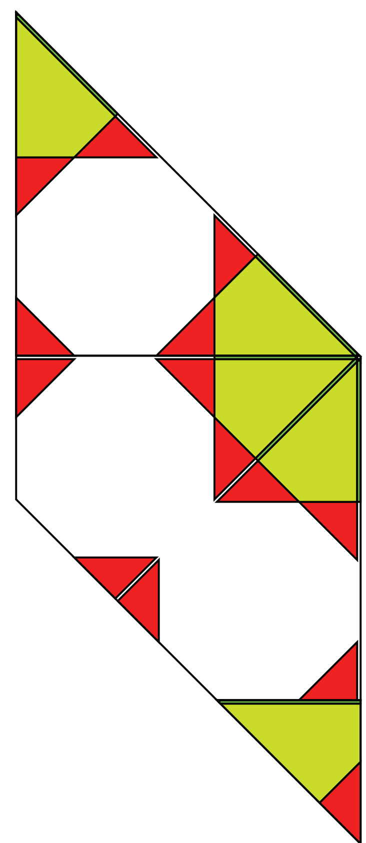


On $\mathrm{I}_{-2k,0}$, $r(s) = \omega^{2k+2}\,(2\,\beta^{2k+1} - s)$

$$\omega = \beta^{-1} = \alpha + 1$$

Renormalization scenario for $s \in I_{-2k,0}$, step 1 : $\mathcal{R}$ induces $\mathcal{B}$





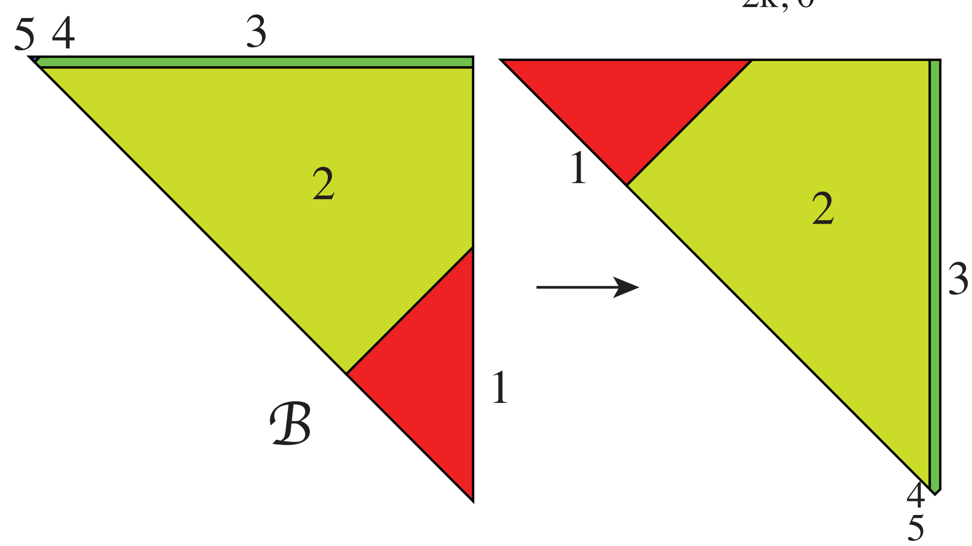
Return paths:

1: $1, (3, 2^2)^4, 3$
2: $1, 3, 2^2, 3$
3: $1, 3, 2^3, 3$
4: $1, 3, 2^4, 3$
5: $1, 3, 2^5, 3$

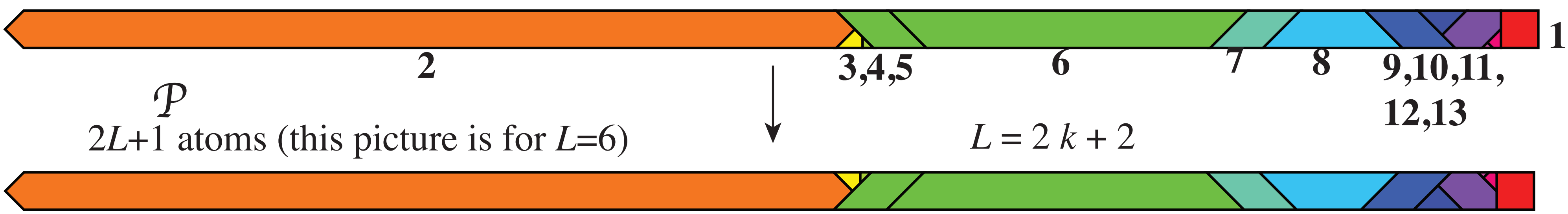
Incidence matrix

$$\begin{pmatrix} 1 & 8 & 5 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 4 & 2 \\ 1 & 5 & 2 \end{pmatrix}$$

Renormalization scenario for $s \in I_{-2k,0}$, step 2 : $\mathcal{B}$ induces $\mathcal{P}$



induces



Return paths:

1: 3

2: 3, 1

3: 3, 1, 4, 1

4: 3, 1, 5, 1

5: 3, 2

6: $3, 2^6$

7: $3, 2^3$

8: $3, 2^2, \phi(1^5), 2$

9: $3, 2^2, \phi(1^2), 2$

$$m = 5, 6, \ldots, L$$

$2m$: $3, 2^2, \phi\big(1^2, \sigma(1)^2, \sigma^2(1)^2 \cdots, \sigma^{L-5}(1)^2, \sigma^{L-4}(1)^5\big), 2$

$2m+1$: $3, 2^2, \phi\big(1^2, \sigma(1)^2, \sigma^2(1)^2 \cdots, \sigma^{L-5}(1)^2, \sigma^{L-4}(1)^2\big), 2$

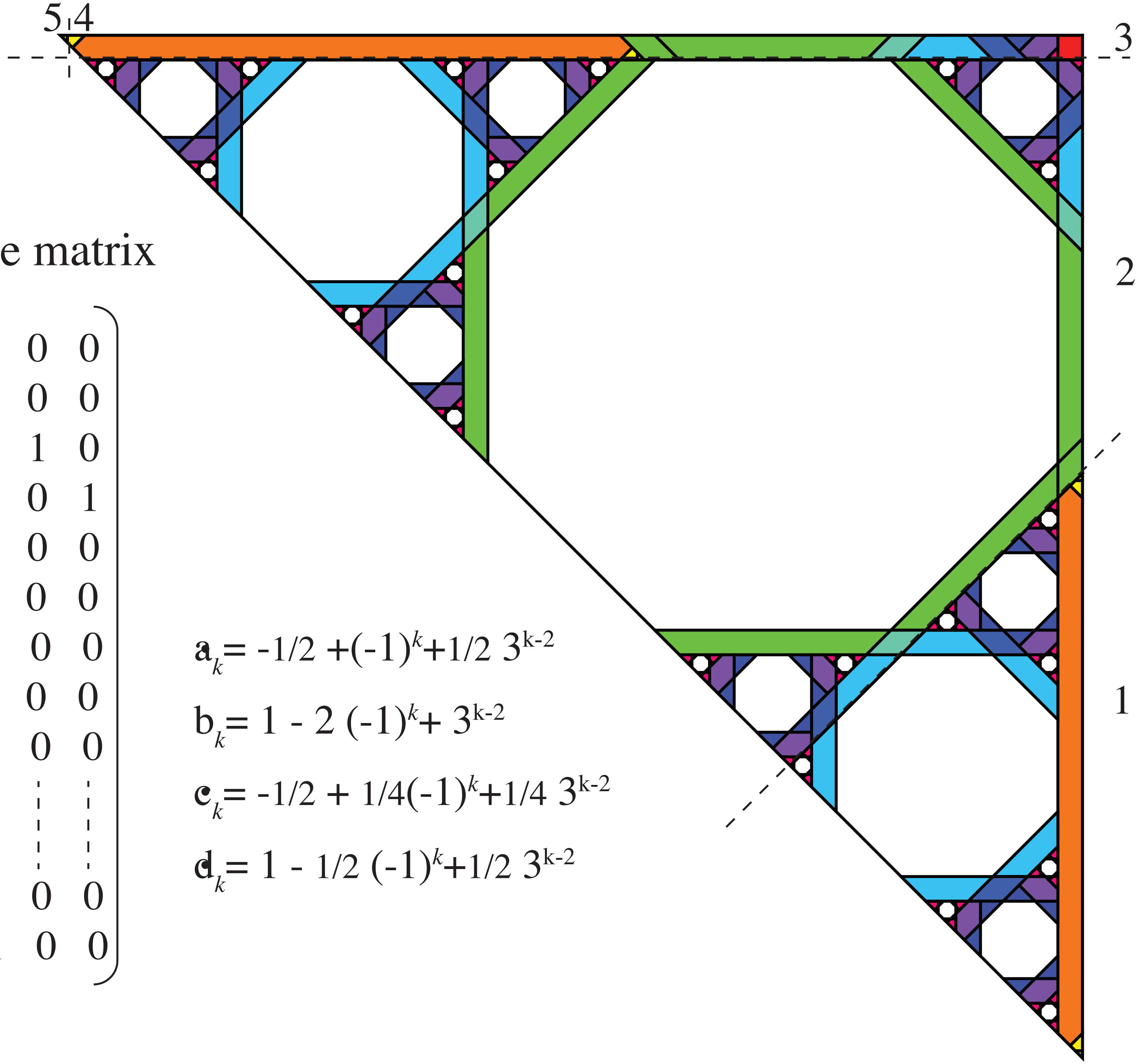
Substitutions: $\phi(1) = (1, 2)$

$\phi(2) = (2, 2, 2)$

$\sigma(1) = (2,1,1)$

$\sigma(2) = (1,1,1)$

$\mathcal{B} \rightarrow \mathcal{P}$ (continued)



Incidence matrix

$$\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
2 & 0 & 1 & 1 & 0 \\
2 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 \\
a_3 & c_3 & 1 & 0 & 0 \\
b_3 & d_3 & 1 & 0 & 0 \\
a_4 & c_4 & 1 & 0 & 0 \\
b_4 & d_4 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_L & c_L & 1 & 0 & 0 \\
b_L & d_L & 1 & 0 & 0
\end{pmatrix}$$

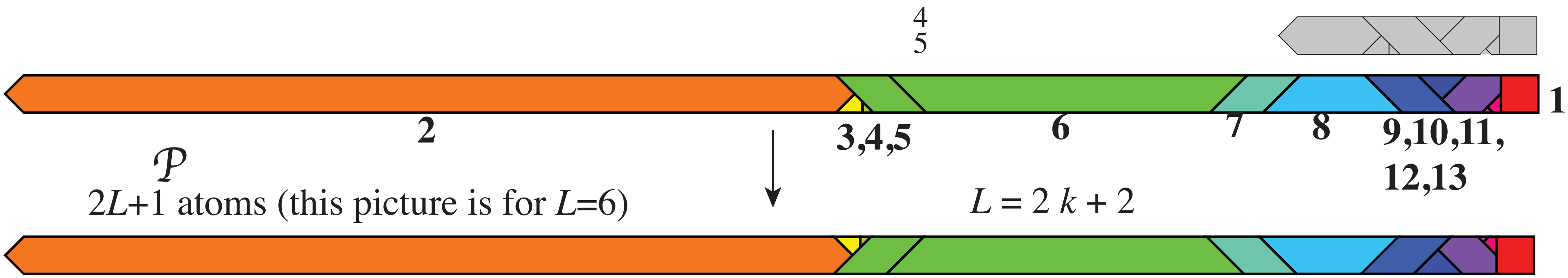
$a_k = -1/2 + (-1)^k + 1/2\ 3^{k-2}$

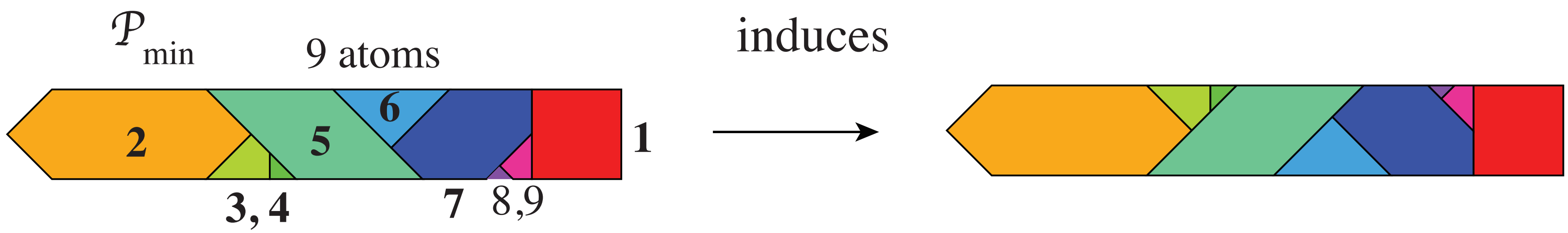
$b_k = 1 - 2\ (-1)^k + 3^{k-2}$

$c_k = -1/2 + 1/4(-1)^k + 1/4\ 3^{k-2}$

$d_k = 1 - 1/2\ (-1)^k + 1/2\ 3^{k-2}$

Step 3: $\mathcal{P}$ induces $\mathcal{P}_{\min}$





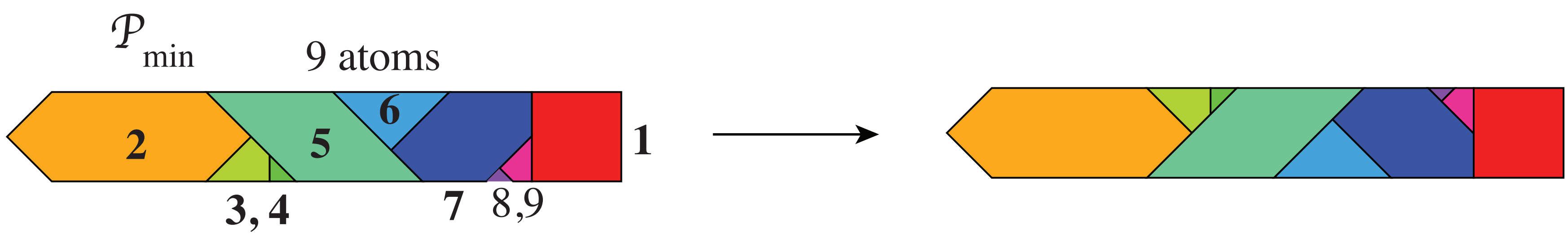
Return paths

1: 1
2: 2 L - 4
3: 2L-4, 2L-5, . . . 6, 5, 3, 5, 6, . . . , 2L-4
4: 2L-4, 2L-5, 2L-5, 2L-4
5: 2L-3
6: 2L-2
7: 2L-1
8: 2L
9: 2L+1

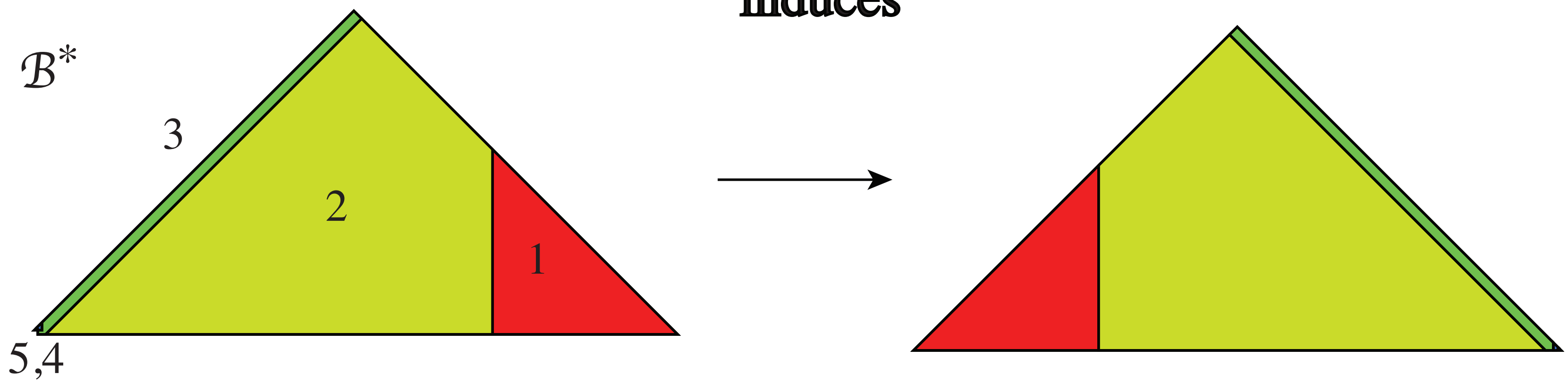
Incidence matrix

$$\left(\begin{array}{cccccccccc|ccccc}
1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & & & \mathbf{0} & & \\
0 & 0 & 1 & 0 & 2 & 2 & \cdots & 2 & 2 & 2 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 & 2 & & & & & \\
\hline
 & & & & & & & & & & 1 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 1 & 0 & 0 & 0 \\
 & & & & \mathbf{0} & & & & & & 0 & 0 & 1 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

And finally, step 4: $\mathcal{P}_{\min}$ induces $\mathcal{B}^*$



induces



Return paths

1: $8,7^2,9,7,6,(5,3,5,6)^6,7,9,7,7$

2: $8,7^2,9,7,6,(5,3,5,6)^3,7,9,7,7$

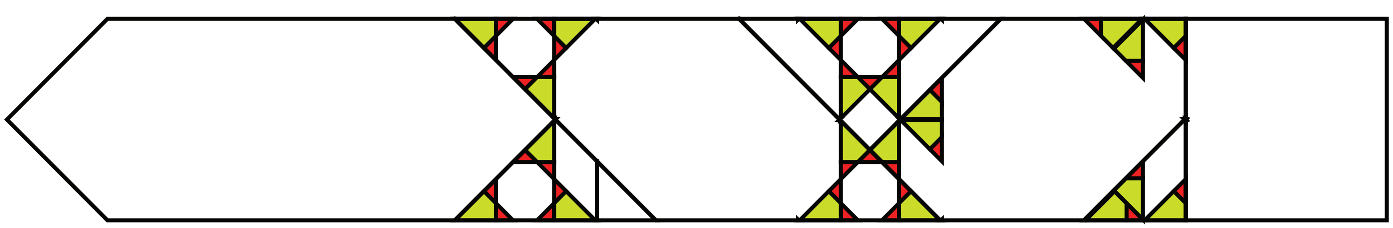
3: $8,7^2$

4: $8,7,6,5,3,5,6,7,9^3,7,6,5,3,5,6,7$

5: $8,7,6,5,3,5,6,7,9^6,7,6,5,3,5,6,7$

Incidence matrix

$$\begin{pmatrix} 0 & 0 & 6 & 0 & 12 & 7 & 6 & 1 & 2 \\ 0 & 0 & 3 & 0 & 6 & 4 & 6 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 & 4 & 4 & 4 & 1 & 3 \\ 0 & 0 & 2 & 0 & 4 & 4 & 4 & 1 & 6 \end{pmatrix}$$



Cumulative incidence matrix for $\mathcal{B} \to \mathcal{P} \to \mathcal{P}_{\min} \to \mathcal{B}^*$

$$\mathbf{A}_{\mathcal{B}\mathcal{B}^*} = \mathbf{A}_{\mathcal{P}_{\min}\mathcal{B}^*} \cdot \mathbf{A}_{\mathcal{P}\mathcal{P}_{\min}} \cdot \mathbf{A}_{\mathcal{B}\mathcal{P}}$$

$$= \mathbf{A} + \mathbf{B}\,L + \mathbf{C}\,3^L$$

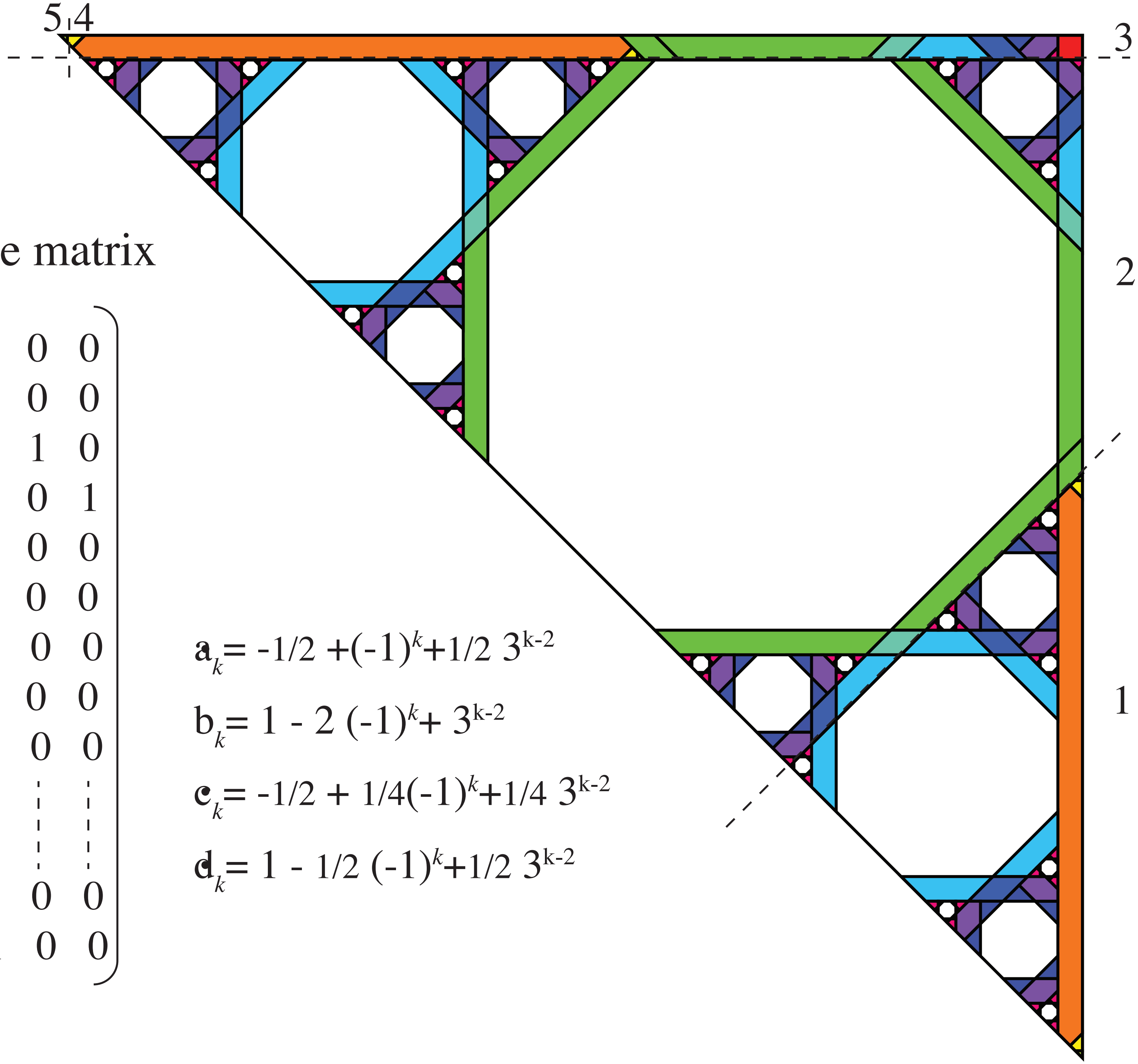
$$\mathbf{A} = \begin{pmatrix} 63/2 & -81 & -62 & 6 & 0 \\ 51/4 & -69/2 & -26 & 3 & 0 \\ -1 & 2 & 3 & 0 & 0 \\ 225/36 & -37/2 & -14 & 2 & 0 \\ 11/2 & -17 & -11 & 2 & 0 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} -12 & 24 & 24 & 0 & 0 \\ -6 & 12 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -4 & 8 & 8 & 0 & 0 \\ -4 & 8 & 8 & 0 & 0 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 25/54 & 25/27 & 0 & 0 & 0 \\ 35/108 & 35/54 & 0 & 0 & 0 \\ 2/27 & 4/27 & 0 & 0 & 0 \\ 11/36 & 11/18 & 0 & 0 & 0 \\ 7/18 & 7/9 & 0 & 0 & 0 \end{pmatrix}$$

Cumulative return times

$$\mathbf{A}_{\mathcal{B}\mathcal{B}^*} \begin{pmatrix} 14 \\ 5 \\ 6 \\ 7 \\ 8 \end{pmatrix} = \begin{pmatrix} -294 \\ -129 \\ 126/9 \\ -75 \\ -60 \end{pmatrix} + \begin{pmatrix} 96 \\ 48 \\ 0 \\ 32 \\ 24 \end{pmatrix} L + \begin{pmatrix} 100 \\ 70 \\ 16 \\ 66 \\ 84 \end{pmatrix} 3^{L-2}$$

$\mathcal{B} \rightarrow \mathcal{P}$ (continued)



Incidence matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ a_3 & c_3 & 1 & 0 & 0 \\ b_3 & d_3 & 1 & 0 & 0 \\ a_4 & c_4 & 1 & 0 & 0 \\ b_4 & d_4 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_L & c_L & 1 & 0 & 0 \\ b_L & d_L & 1 & 0 & 0 \end{pmatrix}$$

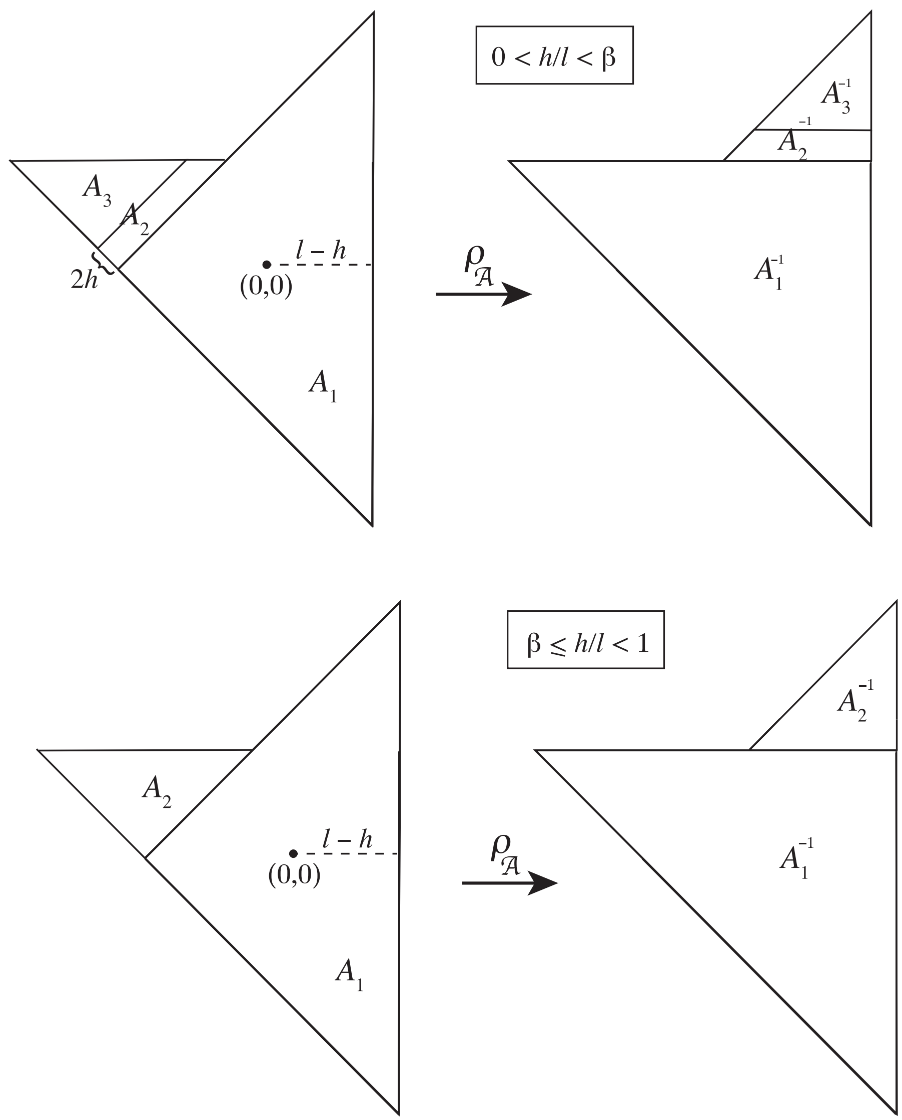
$a_k = -1/2 + (-1)^k + 1/2\ 3^{k-2}$

$b_k = 1 - 2\,(-1)^k + 3^{k-2}$

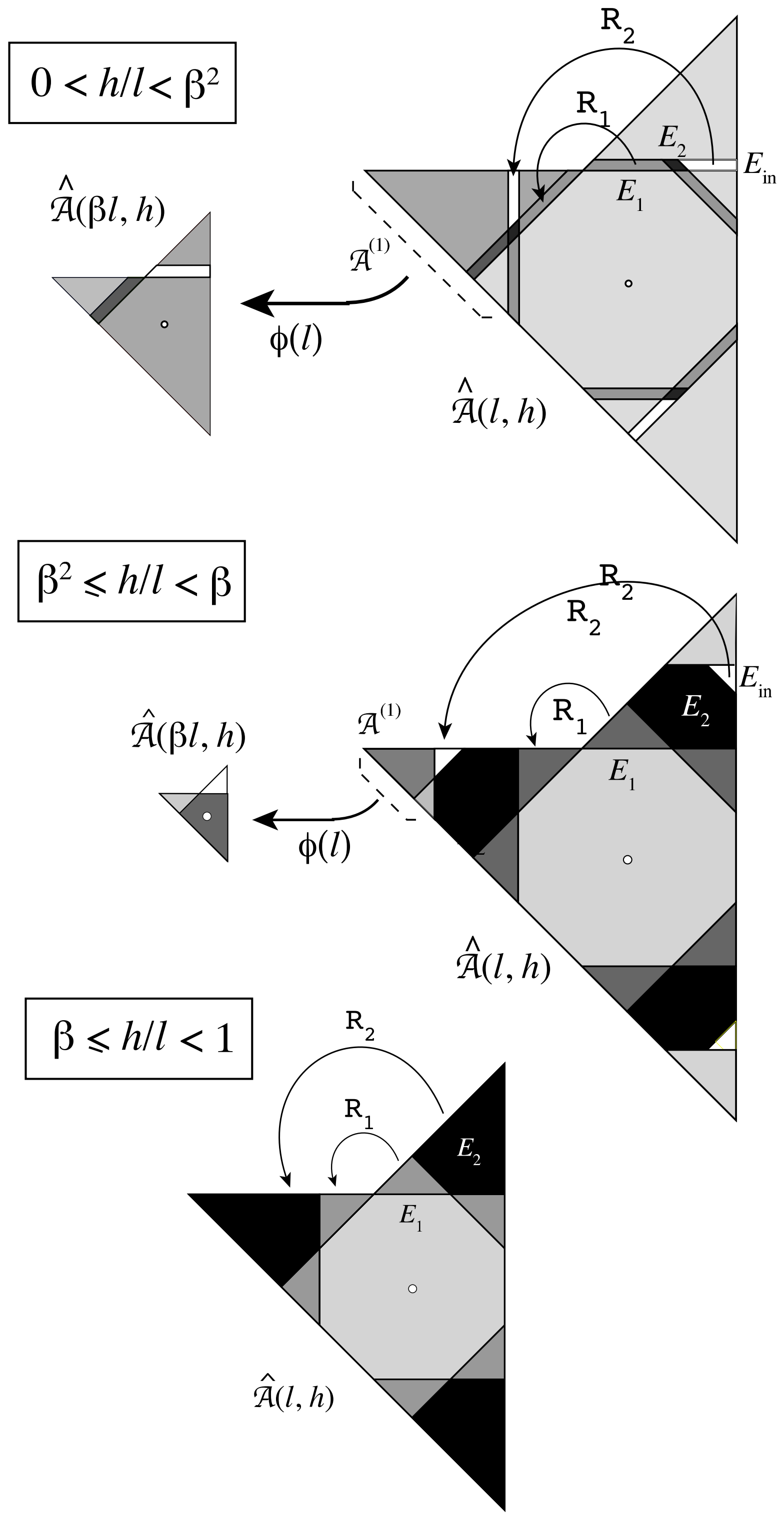
$c_k = -1/2 + 1/4(-1)^k + 1/4\ 3^{k-2}$

$d_k = 1 - 1/2\,(-1)^k + 1/2\ 3^{k-2}$

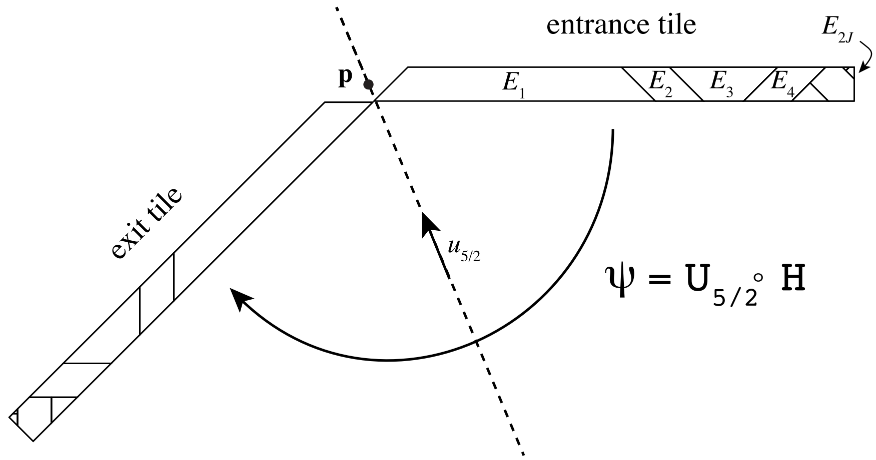
Arrowhead Map



Single-step transfer paths for three parameter ranges



Arrowhead Transfer Map



Partial incidence matrices (visits to tiles 1 and 3 along transfer orbits):

$$N(E_{2k-1}) = \begin{pmatrix} -\frac{1}{2} - (-1)^k + \frac{3}{2}3^k \\ -\frac{1}{2} + (-1)^k + \frac{1}{2}3^k \end{pmatrix},$$

$$N(E_{2k}) = \begin{pmatrix} -\frac{1}{2} - \frac{1}{4}(-1)^k + \frac{3}{4}3^k \\ -\frac{1}{2} + \frac{1}{4}(-1)^k + \frac{1}{4}3^k \end{pmatrix}$$

with $k = 1, \ldots, J(h,l)$.

Simplest self-similar models: s a renormalization fixed point

$$s = r(s) = \omega^L(2\,\beta^{L-1} - s)$$

$$s = \frac{2\omega}{1+\omega^L}$$

Spatial scaling factor $\kappa(L)$: $\mathcal{B}(l\,,s\,)$ induces $\mathcal{B}(\beta^L l\,,s\,)$, and so

$$\kappa(L) = \beta^L$$

Asymptotic temporal scaling factor:

$$\eta(L) = \left|\text{largest eigenvalue of } A_{\mathcal{B}\mathcal{B}^*}\right|$$

$$= \left|\text{largest root of } x^3 + a\,x^2 + b\,x + c\right|$$

$a = -1-10 \times 3^{L-2}$

$b = -56 + 253 \times 3^{L-2} - 72 \times 3^{L-2}\,L$,

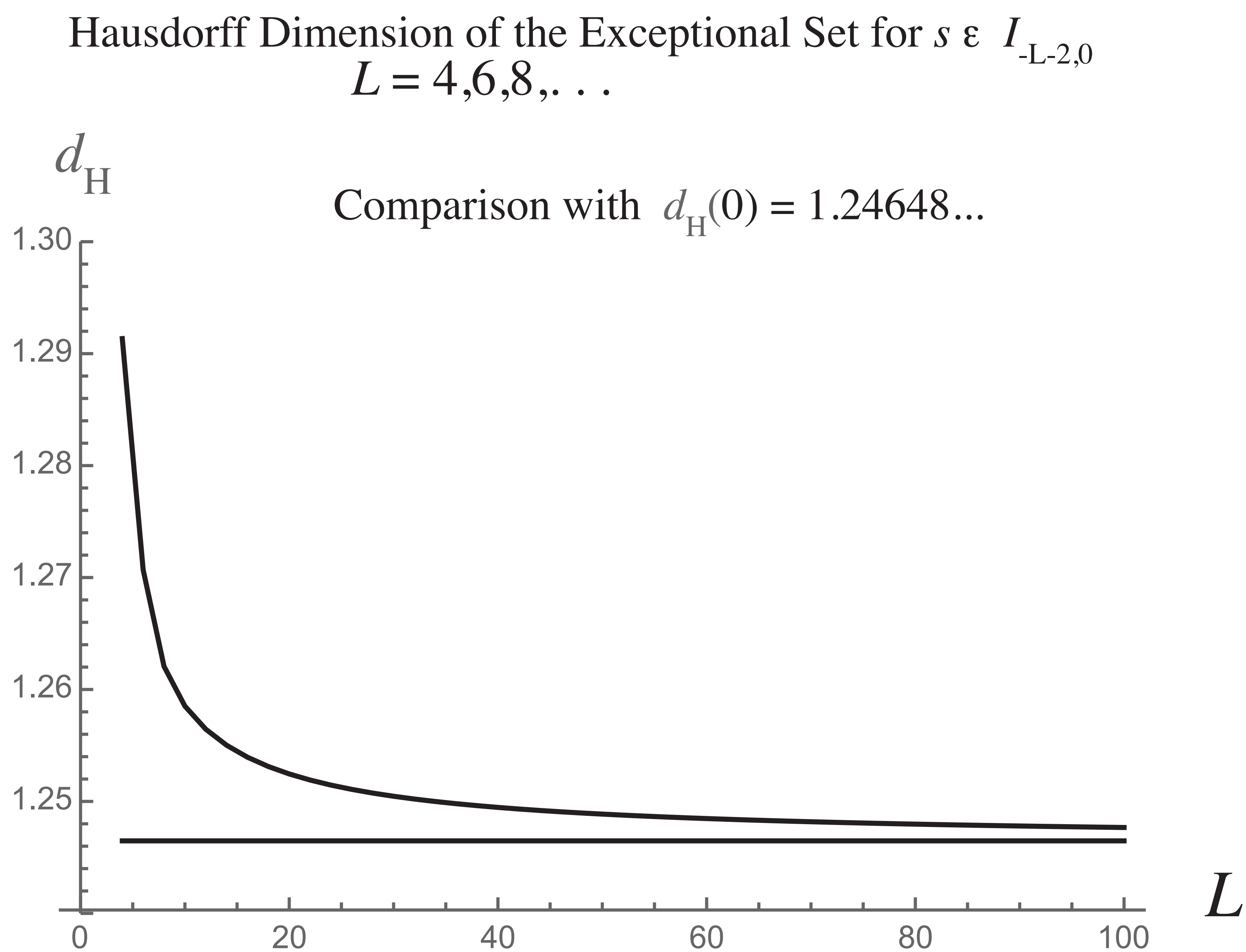
$c = 29 \times 3^{L-2}$

$$= \left(-a + (a^2 - 3b)\,d^{-1/3} + d^{1/3}\right)/3$$

$$2\,d = -2\,a^3 + 9ab - 27c + 3\sqrt{3(-a^2b^2 + 4\,b^3 + 4\,a^3\,c - 18\,abc + 27\,c^2)}$$

Hausdorff dimension of the exceptional set:

$$d_H(L) = -\frac{\operatorname{Log}\eta(L)}{\operatorname{Log}\kappa(L)}$$


Hausdorff Dimension of the Exceptional Set for s ε $I_{-L-2,0}$
L = 4,6,8,. . .
d_H
Comparison with $d_H(0) = 1.24648...$
1.30
1.29
1.28
1.27
1.26
1.25
0
20
40
60
80
100
L