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Entropy dimension for zero entropy systems

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Entropy

- Topological dynamical system (TDS)
  (X, T)—X is a compact metric space with metric d and $T : X \to X$ is continuous.

Definition (topological entropy)

$$h_{top}(X, T) = \sup_{\mathcal{U}} \lim_{n\to\infty} \frac{1}{n} \log N(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}).$$

- Exponential growth:

$$N(\vee_{i=0}^{n-1} T^{-i}\mathcal{U}) \sim e^{nh}$$

- Independence:

  If $\mathcal{U} = \{A_1^c, A_2^c, \cdots, A_k^c\}$ and $h_{top}(T,\mathcal{U}) > 0$, then $\exists W = \{t_1, t_2, \cdots, t_m\} \subset [1, n], d > 0$ with $m = |W| > dn$ such that

$$\bigcap_{i=1}^{m} T^{-t_i} A_{s_i} \neq \emptyset \text{ for any } s_i \in \{1, \cdots, k\}.$$

- If $h_{top}(X, T)$ is finite, then $h_{top}(X, T) = h_{top}(T,\mathcal{U})$ when $\mathcal{U}$ is a generating open cover.

Entropy

- Measure-theoretic dynamical system (MDS) : $(X, \mathcal{B}, \mu, T)$—$(X, \mathcal{B}, \mu)$ is a probability space and $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is measurable and preserves μ.

Definition (measure-theoretic entropy)

$$h_\mu(X, T) = \sup_P \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i} P\right).$$

Entropy

- Exponential growth:

$$H_\mu(\vee_{i=0}^{n-1} T^{-i} P) \sim nh$$

$$\mu(P_n(x)) \sim e^{-nh}, \text{ for } \mu \text{ a.e.} x \in X.$$

- Independence:

$$\text{a Bernoulli factor with entropy } \tilde{h}, \text{ for any } \tilde{h} < h.$$

- If $h_\mu(X, T)$ is finite, then $h_\mu(X, T) = h_\mu(T, P)$ when P is a generating partition.

Zero entropy system

- a wide range of systems:
  rotations, some of Toeplitz systems, distal systems, special flows over rotations, horocycle flows...
- different levels of complexities with subexponential growth rate: $\sim p(n)2^{n^\alpha}$
- zero entropy systems are generic
- "deterministic" systems are not really deterministic

Invariants and entropy type invariants for zero entropy systems

- sequential entropy (Kusnirenko, Goodman)
- scaling entropy (Vershik)
- measure-theoretic complexity (Ferenczi), topological complexity (Blanchard)
- maximal pattern complexity (Kamae), maximal pattern entropy (Huang, Shao and Ye)
- slow entropy (Katok and Thouvenot)

Invariants and entropy type invariants for zero entropy systems

- Entropy dimension(Carvalho, 1997)
- Entropy dimension for measure preserving systems (Park and Ferenczi)
- Entropy dimension via independence and sequence(Dou, Huang and Park)
- Entropy dimension via dimensional topological entropy (Ma, Kuang and Li)

Definition ((upper) topological entropy dimension)

$$\overline{D}(T,\alpha,\mathcal{U}) = \limsup_{n\to\infty} \frac{\log \mathcal{N}(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U})}{n^{\alpha}}.$$

$$\begin{aligned}\overline{D}(T,\mathcal{U}) &= \inf\{\alpha \geq 0 : \overline{D}(T,\alpha,\mathcal{U}) = 0\} \\ &= \sup\{\alpha \geq 0 : \overline{D}(T,\alpha,\mathcal{U}) = \infty\}.\end{aligned}$$

$$\overline{D}(X,T) = \sup_{\mathcal{U}} \overline{D}(T,\mathcal{U})$$

Similarly, we have the definitions of lower entropy dimensions, $\underline{D}(T,\mathcal{U})$ and $\underline{D}(X,T)$.

- $\overline{D}(X, T) = \sup_{\mathcal{W}} \overline{D}(T, \mathcal{W})$, where $\mathcal{W}$ takes over open covers of X with two non-dense open sets.
- $\overline{D}(X_1 \times X_2, T_1 \times T_2) = \max\{\overline{D}(X_1, T_1), \overline{D}(X_2, T_2)\}$.
- If (X_2, T_2) is a factor of (X_1, T_1), then

$$\overline{D}(X_2, T_2) \leq \overline{D}(X_1, T_1) \text{ and } \underline{D}(X_2, T_2) \leq \underline{D}(X_1, T_1).$$

Let $S = \{s_1 < s_2 < \cdots\} \subset \mathbb{N}$, $\tau \geq 0$, we define

$$\overline{D}(S,\tau) = \limsup_{n\to\infty} \frac{n}{(s_n)^\tau} \text{ and } \underline{D}(S,\tau) = \liminf_{n\to\infty} \frac{n}{(s_n)^\tau}.$$

- upper dimension of S:

$$\begin{aligned}\overline{D}(S) &= \inf\{\tau \geq 0 : \overline{D}(S,\tau) = 0\} \\ &= \sup\{\tau \geq 0 : \overline{D}(S,\tau) = \infty\}.\end{aligned}$$

- for $S = \{n^2\}$, $\overline{D}(S) = \frac{1}{2}$.

Similarly, we have definition of $\underline{D}(S)$.

- entropy generating sequence (S):

$$\liminf_{n\to\infty}\frac{1}{n}\log\mathcal{N}\Big(\bigvee_{i=1}^{n}T^{-s_i}\mathcal{U}\Big)>0.$$

- positive entropy sequence (S):

$$h^S_{\mathrm{top}}(T,\mathcal{U}):=\limsup_{n\to\infty}\frac{1}{n}\log\mathcal{N}\Big(\bigvee_{i=1}^{n}T^{-s_i}\mathcal{U}\Big)>0.$$

Definition

$$\overline{D}_e(T,\mathcal{U}) = \sup_S \overline{D}(S), \underline{D}_e(T,\mathcal{U}) = \sup_S \underline{D}(S),$$

where S runs over all entropy generating sequences for $\mathcal{U}$.

$$\overline{D}_p(T,\mathcal{U}) = \sup_S \overline{D}(S), \underline{D}_p(T,\mathcal{U}) = \sup_S \underline{D}(S)$$

where S runs over all positive entropy sequences for $\mathcal{U}$.

$$\overline{D}_e(X,T) = \sup_{\mathcal{U}} \overline{D}_e(T,\mathcal{U}), \quad \overline{D}_p(X,T) = \sup_{\mathcal{U}} \overline{D}_p(T,\mathcal{U}).$$

Similarly, we can define $\underline{D}_e(X,T)$ and $\underline{D}_p(X,T)$.

Theorem

- *Let $A_1, A_2, \cdots, A_k$ be k-pairwise disjoint non-empty closed subsets of a TDS (X, T) $(k \geq 2)$ and $\mathcal{U} = \{A_1^c, A_2^c, \cdots, A_k^c\}$, then*
$$\overline{D}_e(T,\mathcal{U}) = \underline{D}_p(T,\mathcal{U}) = \overline{D}(T,\mathcal{U}).$$
- $$\overline{D}_e(X,T) = \underline{D}_p(X,T) = \overline{D}(X,T).$$
- *If $\mathcal{U}$ is a generating open cover of (X, T), then*
$$\overline{D}_e(T,\mathcal{U}) = \overline{D}(X,T).$$

Let $(x_1, x_2) \in X \times X \setminus \Delta_X$.

- entropy dimension of (x_1, x_2):

$$\overline{D}(x_1, x_2) = \lim_{n\to\infty} \overline{D}(\mathcal{U}_n) \in [0, 1],$$

where $\mathcal{U}_n = \{X \setminus \overline{B(x_1, \frac{1}{n})}, X \setminus \overline{B(x_2, \frac{1}{n})}\}$.

- $\alpha-$pair:

$$\overline{D}(x_1, x_2) = \alpha.$$

- $E^\alpha(X, T)$: collection of $\alpha-$pairs.

Definition

For a TDS (X, T), we call the subset $\{\alpha \geq 0 : E^\alpha(X, T) \neq \emptyset\}$ of $[0, 1]$ the *dimension set* of (X, T) and denote it by $\mathcal{D}(X, T)$.

Theorem

- *If* $0 \notin \mathcal{D}(X, T)$, *then* (X, T) *is topological weakly mixing.*
- *Let* $\pi : (X, T) \to (Y, S)$ *be a factor map between two TDS's. Then* $\mathcal{D}(X, T) \supseteq \mathcal{D}(Y, S)$. *In particularly, dimension set is an invariant under topological conjugacy.*
- *Given* $\alpha \in [0, 1]$, *there exists a TDS* (X, T) *with* $\mathcal{D}(X, T) = \{\alpha\}$.

u.p.e. like properties for topological entropy dimension

Theorem

- *If* $\mathcal{D}(X,T)=\{\alpha\}$, *then any nontrival factor of* (X,T) *has dimension set* $\{\alpha\}$.
- *Let* (Y,S) *be a minimal TDS. If* $\mathcal{D}(X,T)>\mathcal{D}(Y,S)$, *then* (X,T) *is disjoint from* (Y,S).

$J\subseteq X\times Y$ is said to be a *joining* of (X,T) and (Y,S) if J is closed, $T\times S$-invariant with $\pi_X(J)=X$, $\pi_Y(J)=Y$. We say that (X,T) and (Y,S) are *disjoint* if $X\times Y$ contains no proper joining of (X,T) and (Y,S).

One may consider the quantity

$$C(P) = \inf\{\beta : \limsup_{n\to\infty} \frac{1}{n^\beta} H_\mu(\bigvee_{i=0}^{n-1} T^{-i}P) = 0\}.$$

Theorem (Ferenczi and Park 2005)

If there exists a partition P such that $C(P) = \alpha > 0$, then for any $\alpha < \tau < 1$ and $\epsilon > 0$, there exists a partition $\tilde{P}$ such that

$$(1).\ |P - \tilde{P}| < \epsilon,\ \text{and}$$
$$(2).\ C(\tilde{P}) = \tau.$$

Measure-theoretic entropy dimensions

We can define measure-theoretic entropy dimensions through looking the "density" of sequences that generates the complexity.

- entropy generating sequence (S):

$$\liminf_{n\to\infty} \frac{1}{n} H_\mu(\bigvee_{i=1}^{n} T^{-s_i}P) > 0.$$

- positive entropy sequence (S):

$$h_\mu^S(T, P) := \limsup_{n\to\infty} \frac{1}{n} H_\mu(\bigvee_{i=1}^{n} T^{-s_i}P) > 0.$$

Measure-theoretic entropy dimensions

Definition

$$\overline{D}_\mu^e(T,P) = \sup_S \overline{D}(S), \underline{D}_\mu^e(T,P) = \sup_S \underline{D}(S),$$

where S runs over all entropy generating sequences for partition P.

$$\overline{D}_\mu^p(T,P) = \sup_S \overline{D}(S), \underline{D}_\mu^p(T,P) = \sup_S \underline{D}(S)$$

where S runs over all positive entropy sequences for P.

$$\overline{D}_\mu^e(X,T) = \sup_P \overline{D}_\mu^e(T,P), \quad \overline{D}_\mu^p(X,T) = \sup_P \overline{D}_\mu^p(T,P).$$

Similarly, we can define $\underline{D}_\mu^e(X,T)$ and $\underline{D}_\mu^p(X,T)$.

Measure-theoretic entropy dimensions

- $\overline{D}_\mu^p(T, P) = 0$ or 1.
- $$\overline{D}_\mu^e(T, P) = \underline{D}_\mu^p(T, P).$$
- If P is a generating partition (generator), then
$$\overline{D}_\mu^e(X, T) = \overline{D}_\mu^e(T, P) = \underline{D}_\mu^p(T, P) = \underline{D}_\mu^p(X, T).$$

Definition

$$\overline{D}_\mu(T, P) = \overline{D}_\mu^e(T, P) = \underline{D}_\mu^p(T, P),$$
$$\overline{D}_\mu(X, T) = \overline{D}_\mu^e(X, T) = \underline{D}_\mu^p(X, T).$$

Measure-theoretic entropy dimensions

- For any $0 \leq m \leq n$, $\overline{D}_\mu(T, P) = \overline{D}_\mu(T, \bigvee_{i=m}^{n} T^{-i}P)$.
- $\overline{D}_\mu(T, P \vee Q) = \max\{\overline{D}_\mu(T, P), \overline{D}_\mu(T, Q)\}$.
- $$\overline{D}_\mu(X, T) = \sup_P \overline{D}_\mu(T, P),$$
  where P runs over measurable partitions with 2 elements.
- $$\overline{D}_{\mu\times\nu}(X \times Y, T \times S) = \max\{\overline{D}_\mu(X, T), \overline{D}_\nu(Y, S)\}.$$

Definition (dimension set)

$$Dims_\mu(X,T) = \{\overline{D}_\mu(T,\{A, X\setminus A\}) : A \in \mathcal{B} \text{ and } 0 < \mu(A) < 1\}.$$

Theorem

- *Let $(X,\mathcal{B},\mu,T)$ be an invertible MDS and $(Y,\mathcal{D},\nu,S)$ be an ergodic MDS. If $Dims_\mu(X,T) > \overline{D}_\nu(Y,S)$, then $(X,\mathcal{B},\mu,T)$ is disjoint from $(Y,\mathcal{D},\nu,S)$.*
- *Let $\pi : (X,\mathcal{B},\mu,T) \to (Y,\mathcal{D},\nu,S)$ be a factor map between two MDS's. Then $Dims_\mu(X,T) \supseteq Dims_\nu(Y,S)$. In particular, the dimension set is invariant under measurable isomorphism, and so is the entropy dimension.*

K-like properties for measure-theoretic entropy dimension

- For $\tau \in [0,1)$, define the σ-algebra

$$P_\mu^\tau(T) := \{A \in \mathcal{B} : \overline{D}_\mu(T, \{A, X \setminus A\}) \leq \tau\}.$$

- $P_\mu^\tau(T)$ admits the maximal factor with entropy dimension no more than τ.
- $P_\mu^{\tau_1}(T) \subseteq P_\mu^{\tau_2}(T) \subseteq P_\mu(T)$ for any $0 \leq \tau_1 \leq \tau_2 < 1$.

$P_\mu(T) = \{A \in \mathcal{B} : h_\mu(T, \{A, X \setminus A\}) = 0\}$ is Pinsker σ-algebra. It admits the maximal factor with zero entropy and $P_\mu(T) = \{X, \emptyset\}$ for K-system.

Examples

We denote by $\lfloor a \rfloor$ the largest integer not greater than a. For $\tau \in [0,1]$ and $n \geq 2$, let

$$X_{n,\tau} = \{x = (x_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}} : \\ x_j = 0 \text{ for } j \notin \{i\lfloor n^{1-\tau}\rfloor : i = 0,1,\cdots,\lfloor n^{\tau}\rfloor\}\}$$

Let

$$X_\tau = \overline{\cup_{i\in\mathbb{Z}} \cup_{n=1}^{\infty} \sigma^i X_{n,\tau}}.$$

Then $\underline{D}(X,\sigma) = \overline{D}(X,\sigma) = \tau$ and $\overline{D}_\mu(X,\sigma) = 0$ for any invariant measure μ.

Examples

Cassaigne 2003:
Define inductively the substitution $\psi : \mathbb{N}^* \to \{0,1\}^*$ and the family $(x_k)_{k\in\mathbb{N}}$ of prefixes of the dyadic valuation word $\mathbf{v}$ as follows:
a) $\psi(0) = 0, \psi(1) = 1$;
b) x_k is the longest prefix of $\mathbf{v}$ such that

$$|\psi(x_k)| \le \max(\varphi^{-1}(k+1) - \varphi^{-1}(k) - 1, 0);$$

c) for all $j \ge 1$, let

$$\psi(2j) = \psi(x_{\lfloor \log j \rfloor})\mathbf{0}\psi(j) \text{ and } \psi(2j+1) = \psi(x_{\lfloor \log j \rfloor})1\psi(j).$$

Let $\mathbf{u} = \psi(\mathbf{v})$.
A^* the collection of finite or infinite words over A. $\varphi(t) = t^\tau$, $(0 < \tau < 1)$, for example.

Examples

- $\mathbf{u}$ is uniformly recurrent. Let $X = \overline{\{\sigma^i \mathbf{u} : i \in \mathbb{Z}\}}$, then (X, σ) is minimal.
- $C(n)$, complexity function of $\mathbf{u}$, is of $p(n)2^\tau$.
- $D(X, \sigma) = \tau$.
- (X, σ) is uniquely ergodic and $\overline{D}_\mu(X, \sigma) = 0$. (Ahn-Dou-Park, 2010)
- If we replace c) of Cassaigne by the following c'): c') for all $j \geq 1$, let

$$\psi(2j) = \psi(x_{\lfloor \log j \rfloor})\psi(j) \text{ and } \psi(2j+1) = \psi(x_{\lfloor \log j \rfloor})1\psi(j),$$

then $\mathcal{D}(X, \sigma) = \{\tau\}$. (Dou-Huang-Park, 2011).
such systems are called τ-u.d. systems.

Examples

- There exists a MDS with given entropy dimension $\tau \in [0, 1]$. (Ferenczi-Park, 2007)
- There exists a MDS with $\mathcal{D}_\mu(X, T) = \{\tau\}$. (Dou-Huang-Park, arxiv 2013)
- Using cutting and stacking method, controlling the sizes of repetitive and independent steps.

Examples

- Give an irrational rotation:

$$T_\alpha : T^1 \to T^1 = \mathbb{R}/\mathbb{Z}$$

$$T_\alpha \theta = \theta + \alpha \mod 1$$

- Coding to a sturmian system (Z, σ) for a topological purpose:
  Define a bi-infinite $0 - 1$ sequence

$z = z(\theta) = (\cdots, z_{-1} z_0 z_1 \cdots)$ by

$$z_n = \begin{cases} 1 & \text{if } \theta + n\alpha \, mod \, 1 \in [0, \frac{1}{2}) \\ -1 & \text{if } \theta + n\alpha \, mod \, 1 \in [\frac{1}{2}, 1) \end{cases}.$$

We then let $Z = \overline{\{z(\theta), \theta \in T^1\}}$.

Examples

- Give an invertible Bernoulli system (Y, S) with finite entropy.
- Define the skew-product system $(X = Z \times Y, T)$:

$$T(z, y) = (\sigma z, S^{z_0} y)$$

that is, for any point $(z, y) \in Z \times Y$, we will walk along the trajectory of y: if z_n is 1, we walk forward at time n, otherwise we walk backward.

We call this model a **DETERMINISTIC WALKS in RANDOM SCENES**.

Examples

- For any $\tau \in (0,1)$, there exists α such that
$$\overline{D}(X,T) = \overline{D}_\mu(X,T) = \tau.$$
- $$\mathcal{D}(X,T) = \mathcal{D}_\mu(X,T) = \{0,\tau\}.$$

The choice of α:

$$\alpha = [a_0; a_1, a_2, \cdots], \text{ where } a_1 = 5, \cdots, a_{n+1} = 2\lfloor q_n^{\frac{\tau}{1-\tau}} \rfloor + 1, \cdots.$$