Moebius Function:

on the MISSING LOG-factor

et alia

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THREE DEFINITIONS OF MOEBIUS FUNCTION:

- Multiplicative fct with $\mu(p) = -1$, $\mu(p^k) = 0$ $(k \geqslant 2)$,
- *Convolution inverse of* $\mathbb{1}$,
- Coefficients of $1/\zeta(s)$

We concentrate on the *second* definition.

Often an ***explicit*** angle

***Combinatorial** nature* $\rightsquigarrow$ ***functional analysis** pbs!*

Meissel 1854: $$\sum_{n\leqslant x}\mu(n)[x/n] = 1$$

$$\sum_{n\leqslant x}\mu(n)\sum_{m\leqslant x/n} 1 = \sum_{\ell\leqslant x}\sum_{n|\ell}\mu(n)$$

$$\longrightarrow \quad \sum_{n\leqslant x}\mu(n)\{x/n\} = -1 + x\sum_{n\leqslant x}\frac{\mu(n)}{n}$$

Three aspects:

- **Error term treatment** LHS: $\mu(n)$ contaminated by $\{\frac{x}{n}\}$ $\longrightarrow$ RHS: no contamination!
- **Identity** Is it an Accident or a Feature? *No error term!*
- **Log-factor** Trivial bounds: (LHS: x) / (RHS: $x\log x$),

CONSEQUENCES:

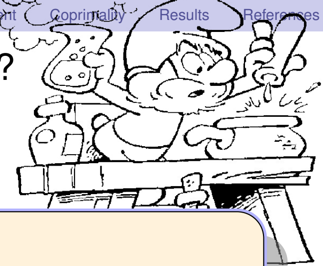
Prime Number Theorem $\rightarrow \sum_{n \leqslant x} \mu(n)\{x/n\} = o(x)$

(Gram 1884): $\left| \sum_{n \leqslant x} \frac{\mu(n)}{n} \right| \leqslant 1$

...

(MacLeod, 1994): $$\sum_{n \leqslant x} \mu(n) \frac{\{x/n\}^2 - \{x/n\}}{x/n} = x \sum_{n \leqslant x} \frac{\mu(n)}{n} - \sum_{n \leqslant x} \mu(n) - 2 + \frac{2}{x}$$

Just curiosities? Better understanding possible?



How to generalize Meissel's proof, I ?

A general Theorem:

Theorem

Let g be a multiplicative function, $|g(p^\nu)| \leqslant K$.

$$\Big|\sum_{n\leqslant x}\frac{g(n)}{n}\Big| \leqslant \frac{3K+3}{\log x}\sum_{n\leqslant x}\frac{|g(n)|+|\mathbb{1}\star g|(n)}{n}$$

Meissel: take $g=\mu$

(S. Selberg, 1954): $$0 \leqslant \sum_{n\leqslant x}\frac{\mu(n)}{2^{\omega(n)}n} \ll 1/\sqrt{\log x}$$

We can rebuild μ with $f\star f$ and $f(n)=\mu(n)/2^{\omega(n)}$ ...

We loose the divisor function!

LANDAU EQUIVALENCE THEOREM (1912)

Theorem

The five propositions are equivalent:

- $\#\{primes \leqslant x\}$ *asymptotic to* $x/\log x$.
- $M(x) = \sum_{n\leqslant x} \mu(n)$ *is* $o(x)$.
- $m(x) = \sum_{n\leqslant x} \mu(n)/n$ *is* $o(1)$.
- $\psi(x) = \sum_{n\leqslant x} \Lambda(n)$ *is asymptotic to* x.
- $\tilde{\psi}(x) = \sum_{n\leqslant x} \Lambda(n)/n$ *is* $\log x - \gamma + o(1)$.

$$\Lambda(n) = \begin{cases} \log p & \text{when } n = p^\nu \\ 0 & \text{else.} \end{cases}$$

$m \to M$ and $\tilde{\psi} \to \psi$: (surprisingly) easy!

Quantitative results? Loss of the prime-number density?

Λ or μ: winning a log-factor is necessary!

Integration by parts ?

$$\tilde\psi(x) - \log x = \frac{\psi(x) - x}{x} + 1 + \int_1^x \frac{(\psi(t) - t)dt}{t^2}$$

From ψ to $\tilde\psi$: false for Beurling integers!

(Diamond & Zhang, 2012)

Beurling integers: the semi-group over primes from $[1, \infty)$

An equivalent statement for Moebius ??

(Axer, 1910): *qualitative aspect*

We need a *quantitative* equivalence

It took me a while!

q-aspect and x-aspect (0R 2002)

Theorem (OR, 2013 + D.Platt, 2016)

There exists $c > 0$ such that, when $x \geqslant 10$:

$$\left|\tilde{\psi}(x) - \log x + \gamma\right| \ll \max_{x \leqslant y \leqslant 2x} \frac{|\psi(y) - y|}{y} + \exp\left(-c\,\frac{\log x}{\log\log x}\right).$$

Same for primes in arithmetic progressions.

Primes $\leqslant x$ require zero-free region up to $\log x$

Hence *(numerically)*

Verifying RH upto H gives control for x upto e^H!!

Sketch of the *now*-proof:

$$\tilde{\psi}(x) - \log x + \gamma = \frac{\psi(x) - x}{x}$$

$$+ \text{ Smoothed version of } \int_x^\infty \frac{(\psi(t) - t)dt}{t^2}$$

controlled by $\max_{x<y\leqslant 2x} |\psi(y) - y|/y$

$+$ Correction from smoothed to unsmoothed

a very convergent sum over the zeroes

Generalizes to primes in arith. prog.

An even better smoothing (Platt & OR, 2016)

But NOT to $M \rightarrow m$?? No zeroes but ...

Theorem (OR, 2016?)

There exists $c > 0$ such that, when $x \geqslant 10$:

$$|m(x)| \ll \max_{x \leqslant y \leqslant 2x} \frac{|M(y)|}{y} + \exp\left(-c\,\frac{\log x}{\log\log x}\right).$$

Yipee!!

And still in progress :)

... But numerically very bad! Bounds for $1/\zeta(s)$ ...

Two consequences:

To motivate further work...

$$\left|\sum_{n\leqslant x}\Lambda(n)/n-\log x+\gamma\right|\leqslant\frac{1}{149\log x}\quad(x\geqslant 23)$$

(Rosser & Schoenfeld, 1962): 2 instead of 149.

$$\left|\sum_{n\leqslant x}\Lambda(n)/n-\log x+\gamma\right|\leqslant\frac{2}{(\log x)^2}\quad(x>1)$$

No ancestors :)

Wanted and final would be: $\varepsilon/(\log x)^2$

Related work by P.Dusart, H.Kadiri, A.Lumley ...

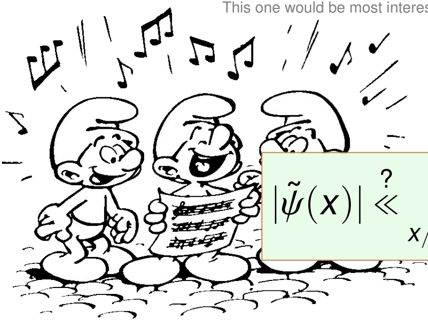
I believe that there exists $A > 1$ such that

$$|m(x)| \overset{?}{\ll} \max_{x/A<y\leqslant xA} |M(y)|/y + x^{-1/4}$$

And in fact

$$|m(x)| \overset{?}{\ll} \max_{x/A<y\leqslant xA} |\psi(y) - y|/y + x^{-1/4}$$

This one would be most interesting since we can express $|\psi(y) - y|$ in terms of the zeros of $\zeta(s)$.



For memory:

$$|\tilde{\psi}(x)| \overset{?}{\ll} \max_{x/A<y\leqslant xA} |\psi(y) - y|/y + x^{-1/4}$$

(Trivially) True under RH
Trivial: with a $\log x$ in front
And with a $\sqrt{\log x}$ in front?

How to generalize Meissel's proof, II ?

Meissel identity reads also

$$\frac{1}{x}\int_1^x M(x/t)\frac{\{t\}}{t}dt = m(x) - \frac{M(x)}{x} - \frac{\log x}{x}$$

(Balazard, 2012) rewrote the first MacLeod identity:

...worked a lot and in between...

$$\frac{1}{x}\int_1^x M(x/t)\frac{(2\{t\}-1)t+\{t\}-\{t\}^2}{t^2}dt = m(x) - \frac{M(x)}{x} - \frac{2}{x} + \frac{2}{x^2}$$

Oldies? Good for Waste-basket?

The situation has been further cleared by F. Daval:

Theorem (Daval, 2016)

Select $h : [0,1] \to \mathbb{C}$, *continuous*, $\int_0^1 h(u)du = 1$. *When* $x \geqslant 1$:

$$\frac{1}{x}\int_1^x M(x/t)\left(1 - \frac{1}{t}\sum_{n\leqslant t} h(n/t)\right)dt$$

$$= m(x) - \frac{M(x)}{x} - \frac{1}{x}\int_{1/x}^1 \frac{h(y)}{y}dy$$

“Riemann integral-remainder”

$h = 1 \to$ Meissel identity

$h = 2t \to$ MacLeod identity

... functional analysis comes in!

Other streams of identities

(Gram, 1884), (MacLeod, 1994), (Balazard, 2012), (OR, 2015)

$$\sum_{n\leqslant x}\frac{\mu(n)}{n}\log\left(\frac{x}{n}\right) - 1 = \frac{6-8\gamma}{3x} - \frac{6-4\gamma}{x^2} + \frac{6-4\gamma}{3x^4}$$

$$(0 \leqslant t^2 h'(t) \leqslant \tfrac{7}{4} - \gamma) \qquad - \frac{1}{x}\int_1^x M(x/t)h'(t)dt$$

$$\rightsquigarrow \sum_{n\leqslant t}\frac{1}{n} - \log t$$

$\log(x/n)$
$\log^2(x/n)$
$\ldots$

$$\to \left|\sum_{n\leqslant x}\frac{\mu(n)}{n}\log(x/n) - 1\right| \leqslant \frac{1}{389\log x}$$

$$(x \geqslant 3\,200)$$

!

THE PROBLEM AT LARGE

GIVEN $F(t) : [1, \infty) \to \mathbb{C}$ e.g. $F(t) = 1$,
FIND H, G & C such that

$$\sum_{n \leqslant x} \frac{\mu(n)}{n} F(x/n) - C\frac{M(x)}{x} = \frac{1}{x}\int_1^x M(x/t)G(t)dt + H(x)$$

$\int_1^\infty |F(t)|dt/t = \infty$

$\int_1^\infty |G(t)|dt/t < \infty$

H smooth and small

With $F = 1$, many solutions!!
With $F = \log t$, many solutions!!

BEGINNING OF A THEORY

$$\int_1^x \left(1 - \frac{1}{t}\sum_{n\leqslant t} h(n/t)\right)dt = \int_0^1 \{ux\}\frac{h(u)}{u}du$$

Aiming at $\int M(x/t)f'(t)dt$.

Given $f : [1,\infty) \to \mathbb{C}$, solve $f(x) = \int_0^1 \{ux\}\frac{h(u)}{u}du$

Given $g : [0,1] \to \mathbb{C}$, solve $g(y) = \int_0^1 \frac{\{u/y\}}{u/y}h(u)du$

Hilbert-Schmidt (hence compact) & contracting!

$$\int_0^1 \frac{\{u/y\}}{u/y}h(u)du = \sum_{n\geqslant 1} \lambda_n \int_0^1 \overline{h(u)}\psi_n(u)du\,\varphi_n(y)$$

$|\lambda_n| \leqslant 1$ ... *Work in progress!*

Shatten class $1+\varepsilon$...

The localization problem

(Daval, 2016): *The basic lemma*

When

$h : [0,1] \mapsto \mathbb{C}$, C^k,

$\int_0^1 h(u)du = 1$, $\qquad h(0) = h'(0) = 0$,

when $3 \leqslant 2i+1 \leqslant k-1$, $\qquad h^{(2i+1)}(0) = 0$,

when $0 \leqslant \ell \leqslant k-2$, $\qquad h^{(\ell)}(1) = 0$,

we have: $$\left|1 - \frac{1}{t}\sum_{n\leqslant t} h(n/t)\right| \ll 1/t^k$$

This class is larger than the earlier ones!

$$\left|\int_1^x M(x/t)\left(1-\frac{1}{t}\sum_{n\leqslant t} h(n/t)\right)dt\right| \leqslant \frac{C_k(h)}{x}\int_1^x M(t)(t/x)^{k-2}dt$$

(Daval, 2016)

$k =$	3	4	5	6	7
$\min_h C_k(h) \leqslant$	1.05	1.44	2.52	5.9	13.2

Best h? Best $C_k(h)$?

$$\left|m(x) - \frac{M(x)}{x}\right| \leqslant \frac{33/13}{x^4}\int_1^x |M(t)|t^3 dt + \frac{19/7}{x}$$

Reversed problem: convolving $M(t)$ with as large a class as possible.

$P(t,\{t\})/t^k$. *Is that all?* $P(t,\{t^2+1\})/t^k$?

From Λ to μ / From ψ to M

$M \to \psi$: NO for Beurling integers! (Zhang, 1987)

$M_{\mathcal{P}}(x) = o(x)$ without $\psi_{\mathcal{P}}(x) \sim x$.

Quantitative Landau: (Kienast, 1926), (Schoenfeld, 1969)

More efficient identities:

$$\sum_{\ell \leqslant x} \mu(\ell) \log^2 \ell = \sum_{d\ell \leqslant x} \mu(\ell) \bigl(\Lambda \star \Lambda(d) - \Lambda(d) \log d\bigr)$$

Analytically?

Only one μ-factor on the RHS:

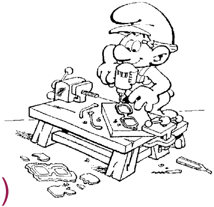
$$\sum_{\ell \leqslant x} \mu(\ell) \log^3 \ell = \sum_{d\ell \leqslant x} \mu(\ell) \bigl(\Lambda \star \Lambda \star \Lambda(d) - 3\Lambda \star (\Lambda \log)(d) + \Lambda(d) \log^2 d\bigr)$$

(OR, 2013)

How to generalize Meissel's proof, III ?

Theorem

When $x, q \geqslant 1$: $$\left|\sum_{\substack{n\leqslant x,\\(n,q)=1}} \frac{\mu(n)}{n}\right| \leqslant 1$$



(Granville & OR, 1996, Lemma 10.2), (Tao, 2010)

... and (Davenport, 1937, Lemma 1)!!

A paper that many of you cite... for another reason!

$$\sum_{n\leqslant x,(n,q)=1} \frac{\mu(n)}{2^{\omega(n)}n}$$ investigated in (S. Selberg, 1954).

Can we do better?

If one removes condition $(d,q)=1$ with Moebius:

$$\sum_{\substack{d\leqslant x,\\(d,q)=1}} \mu(d)/d = \sum_{d\leqslant x}\left(\sum_{\substack{\delta|d,\\ \delta|q}}\mu(\delta)\right)\mu(d)/d$$

$$= \sum_{\delta|q}\mu(\delta)\sum_{\ell\leqslant x/\delta}\mu(\delta\ell)/(\delta\ell)$$

$$= \sum_{\delta|q}\frac{\mu(\delta)^2}{\delta}\sum_{\substack{\ell\leqslant x/\delta,\\(\ell,\delta)=1}}\mu(\ell)/\ell$$

Back to square one!!

Landau's way: $\mathbb{1}_{(d,q)=1}\mu(d)/d = g_q \star (\mu(d)/d)$

But g_q has an infinite support.
And this leads to heavy numerical difficulties :(

$$\sum_{\ell \geqslant 1} g_q(\ell) \sum_{d \leqslant x/\ell} \mu(d)/d \leftarrow$$ Requires explicit estimate also when x/ℓ is small

Workaround:

1. go from μ to λ (Liouville's function),
2. get rid of coprimality by Moebius,
3. go from λ to μ.

Combining steps 1 & 3 $\longrightarrow$ more efficient.

(OR, 2014 and 2015)

Results

A step with Liouville function,

+ rather heavy computations,
and get:

$$\Big|\sum_{\substack{d\leqslant x,\\(d,q)=1}}\mu(d)/d\Big| \leqslant \frac{4q/5}{\phi(q)\log(x/q)} \quad (1\leqslant q<x).$$

... *many similar bounds with* $\mu(d)\log^2(x/d)/d$ *and* $\mu(d)$.

A related problem

(Akhilesh P. & OR, 2016)

Theorem

$$\limsup_{K\to\infty} K \max_q \left| \sum_{\substack{k>K,\\(k,q)=1}} \frac{\mu(k)}{k^2} \right| = 0.$$

The problem occured while studying $G(z)$ (Akhilesh P. & OR, 2016)

What is the best rate of convergence?

$$\limsup_{K\to\infty} K \log K \max_q \left| \sum_{\substack{k>K,\\(k,q)=1}} \frac{\mu(k)}{k^2} \right| \geqslant 1.$$

Can one do better? (RH: $K(\log K)^{1/3-\varepsilon}$)

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