Automatic sequences satisfy
Sarnak’s conjecture II

Clemens Müllner

4. Dec 2016
Automatic Sequences

Definition (Automaton - DFA)

\[ A = (Q, \Sigma = \{0, \ldots, k - 1\}, \delta, q_0, \tau) \]

Example (Thue-Morse sequence)

\[ u = (u_n)_{n \geq 0} = 01101001100101101001011001101001\ldots \]
**Automatic Sequences**

**Definition (Automaton - DFA)**

\[ A = (Q, \Sigma = \{0, \ldots, k - 1\}, \delta, q_0, \tau) \]

**Example (Thue-Morse sequence)**

\[ u = (u_n)_{n \geq 0} = 011010011001011001011001101001 \ldots \]
Automatic Sequences

Definition (Automaton - DFA)

\[ A = (Q, \Sigma = \{0, \ldots, k - 1\}, \delta, q_0, \tau) \]

Example (Thue-Morse sequence)

\[ n = 22 = (10110)_2, \quad u_{22} = 1 \]

\[ u = (u_n)_{n \geq 0} = 011010011001011001011001101001 \ldots \]
Automatic Sequences

Definition (Automaton - DFA)

\[ A = (Q, \Sigma = \{0, \ldots, k - 1\}, \delta, q_0, \tau) \]

Example (Thue-Morse sequence)

\[ n = 22 = (10110)_2, \quad u_{22} = 1 \]

\[ u = (u_n)_{n \geq 0} = 01101001100101100101101101001 \ldots \]
Dynamical System \((X, T)\) related to \(u\)

\[
\begin{align*}
\mathbf{u} &= (u_n)_{n \geq 0} \ldots \text{bounded complex sequence} \\
T \mathbf{u} &= (u_{n+1})_{n \geq 0} \ldots \text{shift operator} \\
X &= \{ T^k(\mathbf{u}) : k \geq 0 \}
\end{align*}
\]

We say that \(u\) satisfies the **Sarnak conjecture** if all sequences \(a = (a_n)_{n \geq 0} \in X\) are orthogonal to \(\mu(n)\).
Dynamical System \((X, T)\) related to \(u\)

\[
\begin{align*}
\mathbf{u} &= (u_n)_{n \geq 0} \ldots \text{bounded complex sequence} \\
Tu &= (u_{n+1})_{n \geq 0} \ldots \text{shift operator} \\
X &= \overline{\{T^k(u) : k \geq 0\}}
\end{align*}
\]

We say that \(u\) satisfies the Sarnak conjecture if all sequences \(a = (a_n)_{n \geq 0} \in X\) are orthogonal to \(\mu(n)\).
Dynamical System \((X, T)\) related to \(u\)

\[
\begin{align*}
\mathbf{u} &= (u_n)_{n \geq 0} \ldots \text{bounded complex sequence} \\
T \mathbf{u} &= (u_{n+1})_{n \geq 0} \ldots \text{shift operator} \\
X &= \{ T^k(\mathbf{u}) : k \geq 0 \}
\end{align*}
\]

We say that \(u\) satisfies the Sarnak conjecture if all sequences \(a = (a_n)_{n \geq 0} \in X\) are orthogonal to \(\mu(n)\).
Dynamical System \((X, T)\) related to \(u\)

\[
\begin{align*}
  u &= (u_n)_{n \geq 0} \ldots \text{bounded complex sequence} \\
  Tu &= (u_{n+1})_{n \geq 0} \ldots \text{shift operator} \\
  X &= \{T^k(u) : k \geq 0\}
\end{align*}
\]

We say that \(u\) satisfies the **Sarnak conjecture** if all sequences \(a = (a_n)_{n \geq 0} \in X\) are orthogonal to \(\mu(n)\).
Theorem 1 (M., 2016)

Every automatic sequence $(a_n)_{n \geq 0}$ fulfills the Sarnak Conjecture.

Theorem 2 (M., 2016)

Let $A = (Q', \Sigma, \delta', q'_0, \tau)$ be a strongly connected DFAO such that $\Sigma = \{0, \ldots, k - 1\}$ and $\delta'(q'_0, 0) = q'_0$. Then the frequencies of the letters for the prime-subsequence $(a_p)_{p \in \mathcal{P}}$ exist, i.e.

$$
dens_{\mathcal{P}}(u, \alpha) = \lim_{N \to \infty} \frac{1}{\pi(N)} \sum_{1 \leq p \leq N} 1_{[u_p = \alpha]}.
$$

Remark: All block-additive (i.e. digital) functions are covered by Theorem 2 and they are ”usually” uniformly distributed.
Theorem 1 (M., 2016)

Every automatic sequence \((a_n)_{n \geq 0}\) fulfills the Sarnak Conjecture.

Theorem 2 (M., 2016)

Let \(A = (Q', \Sigma, \delta', q_0', \tau)\) be a strongly connected DFAO such that \(\Sigma = \{0, \ldots, k - 1\}\) and \(\delta'(q_0', 0) = q_0'\). Then the frequencies of the letters for the prime-subsequence \((a_p)_{p \in \mathbb{P}}\) exist, i.e.

\[
dens_{\mathbb{P}}(u, \alpha) = \lim_{N \to \infty} \frac{1}{\pi(N)} \sum_{1 \leq p \leq N} 1_{[u_p = \alpha]}.
\]

Remark: All block-additive (i.e. digital) functions are covered by Theorem 2 and they are ”usually” uniformly distributed.
Theorem 1 (M., 2016)

Every automatic sequence \((a_n)_{n \geq 0}\) fulfills the Sarnak Conjecture.

Theorem 2 (M., 2016)

Let \(A = (Q', \Sigma, \delta', q'_0, \tau)\) be a strongly connected DFAO such that \(\Sigma = \{0, \ldots, k-1\}\) and \(\delta'(q'_0, 0) = q'_0\). Then the frequencies of the letters for the prime-subsequence \((a_p)_{p \in P}\) exist, i.e.

\[
dens_P(u, \alpha) = \lim_{N \to \infty} \frac{1}{\pi(N)} \sum_{1 \leq p \leq N} 1[u_p = \alpha].
\]

Remark: All block-additive (i.e. digital) functions are covered by Theorem 2 and they are "usually" uniformly distributed.
Lemma

Let \((a_n)_{n \in \mathbb{N}}\) be a sequence that takes values in \(\Delta\). Suppose that for every \(j \geq 1\) and for every function \(g : \Delta^j \to \mathbb{C}\) we have

\[
\sum_{n \leq N} g(a_n + \ell, \ldots, a_n + \ell + j - 1) \mu(n) = o(N),
\]

uniformly for \(\ell \in \mathbb{N}\). Then, \((a_n)_{n \in \mathbb{N}}\) fulfills the Sarnak Conjecture.

Lemma

Suppose that for every automatic sequence \((a_n)_{n \in \mathbb{N}}\) with values in \(\mathbb{C}\)

\[
\sum_{n \leq N} \mu(n) a_{n+r} = o(N),
\]

uniformly for \(r \in \mathbb{N}\). Then Theorem 1 holds.
Lemma

Let \((a_n)_{n \in \mathbb{N}}\) be a sequence that takes values in \(\Delta\). Suppose that for every \(j \geq 1\) and for every function \(g : \Delta^j \rightarrow \mathbb{C}\) we have

\[
\sum_{n \leq N} g(a_{n+\ell}, \ldots, a_{n+\ell+j-1}) \mu(n) = o(N),
\]

uniformly for \(\ell \in \mathbb{N}\). Then, \((a_n)_{n \in \mathbb{N}}\) fulfills the Sarnak Conjecture.

Lemma

Suppose that for every automatic sequence \((a_n)_{n \in \mathbb{N}}\) with values in \(\mathbb{C}\)

\[
\sum_{n \leq N} \mu(n) a_{n+r} = o(N),
\]

uniformly for \(r \in \mathbb{N}\). Then Theorem 1 holds.
Definition (Synchronizing Automaton / Word)

\[ \exists w_0 : \delta(q, w_0) = a \quad \forall q. \]

Example

\[ w_0 = 010. \]
**Definition (Synchronizing Automaton / Word)**

\[ \exists w_0 : \delta(q, w_0) = a \quad \forall q. \]

**Example**

\[ w_0 = 010. \]
Synchronizing Automata

Definition (Synchronizing Automaton / Word)

\[ \exists w_0 : \delta(q, w_0) = a \quad \forall q. \]

Example

\[ w_0 = 010. \]
Recapitulation

Transition Matrix

\[ M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
Recapitulation

Transition Matrix

\[ M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
Recapitulation

Transition Matrix

\[ M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ; 
M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} ; 
M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \]
Recapitulation

Transition Matrix

\[ M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \]

\[ 11 = (102)_3 : \quad M_2 \circ M_0 \circ M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]
Recapitulation

过渡矩阵

\[ q_1 \rightarrow q_2 \leftarrow q_3 \]

\[ M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ; M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} ; M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \]

\[ T(n) := M_{\epsilon_0(n)} M_{\epsilon_1(n)} \cdots M_{\epsilon_{\ell-1}(n)} \]

\[ u(n) = f(T(n)e_1) \]

\[ e_1 = (1 \ 0 \ 0)^T \]
Invertible Automata

**Definition**

An automaton is called invertible if all transition matrices $M_0, \ldots, M_{k-1}$ are invertible and if $M = M_0 + \ldots + M_{k-1}$ is primitive.

$M$ is primitive iff there exists $m \geq 0$ such that for every $a, b \in Q$ exists $w \in \Sigma^m$ such that $\delta(a, w) = b$.

**Remark:**

If the matrix $M = M_0 + \ldots + M_{k-1}$ is primitive then the frequencies $freq(u, a) = \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} 1[u_n = a]$ exist.
**Definition**

An automaton is called invertible if all transition matrices $M_0, \ldots, M_{k-1}$ are invertible and if $M = M_0 + \ldots + M_{k-1}$ is primitive.

$M$ is primitive iff there exists $m \geq 0$ such that for every $a, b \in Q$ exists $w \in \Sigma^m$ such that $\delta(a, w) = b$.

**Remark:**

If the matrix $M = M_0 + \ldots + M_{k-1}$ is primitive then the frequencies

$$freq(u, a) = \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} 1[u_n = a]$$

exist.
Definition

An automaton is called invertible if all transition matrices $M_0, \ldots, M_{k-1}$ are invertible and if $M = M_0 + \ldots + M_{k-1}$ is primitive.

$m$ is primitive iff there exists $m \geq 0$ such that for every $a, b \in Q$ exists $w \in \Sigma^m$ such that $\delta(a, w) = b$.

Remark:
If the matrix $M = M_0 + \ldots + M_{k-1}$ is primitive then the frequencies

$$freq(u, a) = \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} 1[u_n = a]$$

exist.
Suppose that an automatic sequence $u = (u_n)_{n \geq 0}$ is generated by an invertible automaton.

**Theorem [Drmota, Ferenczi + Kulaga-Przymus+Lemanczyk+Mauduit]**

- $u$ is orthogonal to $\mu(n)$.

**Theorem [Drmota]**

- The frequency of each letter of the subsequence $(u_p)_{p \in \mathcal{P}}$ exists.
Recapitulation

Invertible Automata

Results for Invertible Automata

Suppose that an automatic sequence \( u = (u_n)_{n \geq 0} \) is generated by an invertible automaton.

**Theorem [Drmota, Ferenczi + Kulaga-Przymus+Lemanczyk+Mauduit]**

The sequence \( u \) is orthogonal to \( \mu(n) \).

**Theorem [Drmota]**

The frequency of each letter of the subsequence \( (u_p)_{p \in \mathcal{P}} \) exists.
Suppose that an automatic sequence $u = (u_n)_{n \geq 0}$ is generated by an invertible automaton.

**Theorem [Drmota, Ferenczi + Kulaga-Przymus+Lemanczyk+Mauduit]**

$u$ is orthogonal to $\mu(n)$.

**Theorem [Drmota]**

The frequency of each letter of the subsequence $(u_p)_{p \in \mathcal{P}}$ exists.
We call a sequence \((a_n)_{n \geq 0}\) digital if there exists \(m \geq 1\) and \(F : \{0, \ldots, k - 1\}^m \to \mathbb{C}\) such that

\[
a_n = \sum_{i \geq 0} F(\varepsilon_{i+m-1}(n), \ldots, \varepsilon_i(n)).
\]

**Lemma**

Let \((a_n)_{n \geq 0}\) be a digital sequence. Then \((a_n \mod m')_{n \geq 0}\) is an automatic sequence for every \(m' \in \mathbb{N}\).

**Example**

The sum of digits function in base \(k\), \(s_k(n)\) is digital where \(m = 1\) and \(F(x) = x\).
We call a sequence \((a_n)_{n \geq 0}\) digital if there exists \(m \geq 1\) and \(F : \{0, \ldots, k-1\}^m \to \mathbb{C}\) such that

\[
a_n = \sum_{i \geq 0} F(\varepsilon_{i+m-1}(n), \ldots, \varepsilon_i(n)).
\]

**Lemma**

Let \((a_n)_{n \geq 0}\) be a digital sequence. Then \((a_n \mod m')_{n \geq 0}\) is an automatic sequence for every \(m' \in \mathbb{N}\).

**Example**

The sum of digits function in base \(k\), \(s_k(n)\) is digital where \(m = 1\) and \(F(x) = x\).
Digital Sequences

We call a sequence \((a_n)_{n \geq 0}\) digital if there exists \(m \geq 1\) and \(F : \{0, \ldots, k - 1\}^m \to \mathbb{C}\) such that

\[
a_n = \sum_{i \geq 0} F(\epsilon_{i+m-1}(n), \ldots, \epsilon_i(n)).
\]

**Lemma**

Let \((a_n)_{n \geq 0}\) be a digital sequence. Then \((a_n \mod m')_{n \geq 0}\) is an automatic sequence for every \(m' \in \mathbb{N}\).

**Example**

The sum of digits function in base \(k\), \(s_k(n)\) is digital where \(m = 1\) and \(F(x) = x\).
Theorem [Mauduit + Rivat, Tao]
The Rudin-Shapiro Sequence is orthogonal to the Möbius function.
Example (Rudin-Shapiro)

Theorem [Mauduit + Rivat, Tao]
The Rudin-Shapiro Sequence is orthogonal to the Möbius function.
Definition (Naturally Induced Transducer)

Let $A = (Q', \Sigma, \delta', q_0')$ be a strongly connected automata. We call $T_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ a **naturally induced transducer** iff

1. $\exists n_0 \in \mathbb{N} : Q \subseteq (Q')^{n_0}$
2. $T_A$ is synchronizing
3. “attach to each transition $\delta(q, a)$ a permutation $\lambda(q, a)$“.
4. $\delta'(q, a) = \lambda(q, a) \cdot \delta(q, a)$
5. some minimality/technical conditions
Definition (Naturally Induced Transducer)

Let $A = (Q', \Sigma, \delta', q_0')$ be a strongly connected automata. We call $T_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ a naturally induced transducer iff

1. $\exists n_0 \in \mathbb{N} : Q \subseteq (Q')^{n_0}$
2. $T_A$ is synchronizing
3. “attach to each transition $\delta(q, a)$ a permutation $\lambda(q, a)$“.
4. $\delta'(q, a) = \lambda(q, a) \cdot \delta(q, a)$
5. some minimality/technical conditions
Definition (Naturally Induced Transducer)

Let \( A = (Q', \Sigma, \delta', q'_0) \) be a strongly connected automata. We call \( T_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda) \) a **naturally induced transducer** iff

1. \( \exists n_0 \in \mathbb{N} : Q \subseteq (Q')^{n_0} \)
2. \( T_A \) is synchronizing
3. “attach to each transition \( \delta(q, a) \) a permutation \( \lambda(q, a) \)”.
4. \( \delta'(q, a) = \lambda(q, a) \cdot \delta(q, a) \)
5. some minimality/technical conditions
Definition (Naturally Induced Transducer)

Let $A = (Q', \Sigma, \delta', q_0')$ be a strongly connected automata. We call $T_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ a naturally induced transducer iff

1. $\exists n_0 \in \mathbb{N} : Q \subseteq (Q')^{n_0}$
2. $T_A$ is synchronizing
3. “attach to each transition $\delta(q, a)$ a permutation $\lambda(q, a)$“.
4. $\delta'(q, a) = \lambda(q, a) \cdot \delta(q, a)$
5. some minimality/technical conditions
Definition (Naturally Induced Transducer)

Let $A = (Q', \Sigma, \delta', q'_0)$ be a strongly connected automata. We call $T_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ a naturally induced transducer iff

1. $\exists n_0 \in \mathbb{N} : Q \subseteq (Q')^{n_0}$
2. $T_A$ is synchronizing
3. “attach to each transition $\delta(q, a)$ a permutation $\lambda(q, a)$“.
4. $\delta'(q, a) = \lambda(q, a) \cdot \delta(q, a)$
5. some minimality/technical conditions
Definition (Naturally Induced Transducer)

Let \( A = (Q', \Sigma, \delta', q'_0) \) be a strongly connected automata. We call \( T_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda) \) a naturally induced transducer iff

1. \( \exists n_0 \in \mathbb{N} : Q \subseteq (Q')^{n_0} \)
2. \( T_A \) is synchronizing
3. “attach to each transition \( \delta(q, a) \) a permutation \( \lambda(q, a) \)“.
4. \( \delta'(q, a) = \lambda(q, a) \cdot \delta(q, a) \)
5. some minimality/technical conditions
Examples

Example (Synchronizing Automaton)

\[
\begin{array}{c}
\text{start} \rightarrow a \quad b \\
0 \quad 0,1 \quad 1
\end{array}
\]

\[
\begin{array}{c}
\text{start} \rightarrow a \quad b \\
0 \mid \text{id} \quad 0 \mid \text{id} \quad 1 \mid \text{id}
\end{array}
\]
Examples

Example (Synchronizing Automaton)

\[ a \xrightarrow{0} 0,1 \xrightarrow{1} b \]

\[ a \xrightarrow{0 \mid \text{id}} 0 \mid \text{id} \xrightarrow{1 \mid \text{id}} b \]
Examples

Example (Invertible Automaton)

\[ q_1, q_2, q_3 \]

\[ \begin{array}{ccc}
0 & | & id \\
1 & | & (12) \\
2 & | & (123) \\
\end{array} \]
Example (Invertible Automaton)

\[ \begin{array}{c}
q_1 & \xrightarrow{0} & q_2 & \xrightarrow{2} & q_1 \\
q_2 & \xrightarrow{1} & q_3 & \xrightarrow{1} & q_2 \\
q_3 & \xrightarrow{2} & q_3 & \xrightarrow{2} & q_2 \\
q_1 & \xrightarrow{0} & q_1 & \xrightarrow{1} & q_1
\end{array} \]

\[ \begin{array}{l}
0 \mid \text{id} \\
1 \mid (12) \\
2 \mid (123)
\end{array} \]

\[ \begin{array}{c}
\text{start} \rightarrow q_1, q_2, q_3
\end{array} \]
Theorem

For every strongly connected automaton $A$, there exists a naturally induced transducer $T_A$. All other naturally induced transducers can be obtained by changing the order on the elements of $Q$.

Example:
Theorem

For every strongly connected automaton $A$, there exists a naturally induced transducer $T_A$. All other naturally induced transducers can be obtained by changing the order on the elements of $Q$.

Example:
Theorem

For every strongly connected automaton $A$, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of $Q$.

Example:
Theorem

For every strongly connected automaton $A$, there exists a naturally induced transducer $T_A$. All other naturally induced transducers can be obtained by changing the order on the elements of $Q$.

Example:
**Theorem**

For every strongly connected automaton $A$, there exists a naturally induced transducer $T_A$. All other naturally induced transducers can be obtained by changing the order on the elements of $Q$.

**Example:**

\[ q_0' \rightarrow 1, \quad q_1' \rightarrow 0, \quad q_2' \rightarrow 0, \quad q_3' \rightarrow 1, \quad q_4' \rightarrow 0,1 \]
Theorem

For every strongly connected automaton $A$, there exists a naturally induced transducer $T_A$. All other naturally induced transducers can be obtained by changing the order on the elements of $Q$.

Example:
Theorem
For every strongly connected automaton $A$, there exists a naturally induced transducer $T_A$. All other naturally induced transducers can be obtained by changing the order on the elements of $Q$.

Example:
Theorem

For every strongly connected automaton $A$, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of $Q$.

Example:
Example (Digital Sequence)

„Generic Example“: \( k = 2, m = 3, m' = 3 \)
\[ F(010) = 1, F(110) = 2, F(101) = 1 \]

- Every word of length \( m - 1 \) is synchronizing.
- The group generated by the permutations is cyclic.
"Generic Example": $k = 2$, $m = 3$, $m' = 3$

$F(010) = 1$, $F(110) = 2$, $F(101) = 1$

Every word of length $m - 1$ is synchronizing.

The group generated by the permutations is cyclic.
Example (Digital Sequence)

„Generic Example“: $k = 2$, $m = 3$, $m' = 3$

$F(010) = 1$, $F(110) = 2$, $F(101) = 1$

- Every word of length $m - 1$ is synchronizing.
- The group generated by the permutations is cyclic.
Motivation

Example (Digital Sequence)

„Generic Example“: \( k = 2, m = 3, m' = 3 \)
\( F(010) = 1, F(110) = 2, F(101) = 1 \)

- Every word of length \( m - 1 \) is synchronizing.
- The group generated by the permutations is cyclic.
Definition

Denote by

\[ T(q, w_1 \ldots w_r) := \lambda(q, w_1) \circ \lambda(\delta(q, w_1), w_2) \circ \ldots \circ \lambda(\delta(q, w_1 \ldots w_{r-1}), w_r). \]

Lemma

Let \( A \) be a strongly connected automaton and \( T_A \) a naturally induced transducer. Then,

\[ \delta'(q_0', w) = \pi_1(T(q_0, w) \cdot \delta(q_0, w)) \]

holds for all \( w \in \Sigma^* \).
Definition

Denote by

\[ T(q, w_1 \ldots w_r) := \lambda(q, w_1) \circ \lambda(\delta(q, w_1), w_2) \circ \ldots \circ \lambda(\delta(q, w_1 \ldots w_{r-1}), w_r). \]

Lemma

Let \( A \) be a strongly connected automaton and \( T_A \) a naturally induced transducer. Then,

\[ \delta'(q'_0, w) = \pi_1(T(q_0, w) \cdot \delta(q_0, w)) \]

holds for all \( w \in \Sigma^* \).
Are some naturally induced transducers better than others?

(Oversimplified) Example

- Naturally Induced Transducer
- Properties of naturally induced transducers
Are some naturally induced transducers better than others?

(Oversimplified) Example

\[
\begin{align*}
\text{a} & \xrightarrow{0,1} \text{c} \\
\text{b} & \xrightarrow{0,1} \text{d} \\
\text{c} & \xrightarrow{1} \text{a} \\
\text{d} & \xrightarrow{1} \text{b}
\end{align*}
\]

\[
\begin{align*}
\text{a, b} & \xrightarrow{0 \mid \text{id}} \text{c, d} \\
\text{c, d} & \xrightarrow{1 \mid \text{id}} \text{a, b} \\
\text{a, b} & \xrightarrow{0 \mid (12)} \text{d, c} \\
\text{d, c} & \xrightarrow{1 \mid (12)} \text{a, b}
\end{align*}
\]
Are some naturally induced transducers better than others?

(Oversimplified) Example

\[
\begin{align*}
\text{a} & \quad \text{b} & \quad \text{a, b} & \quad \text{a, b} \\
0 & \quad 0 & \quad 0 | \text{id} & \quad 0 | \text{id} \\
| & \quad | & \quad | & \quad | \\
1 & \quad 1 & \quad 1 | \text{id} & \quad 1 | (12)
\end{align*}
\]
Are some naturally induced transducers better than others?

(Oversimplified) Example
All elements of $\Delta$ appear as values of $T(q_0, .)$ for "good" naturally induced transducer.

Do all elements of $\Delta$ appear simultaneously as values of $T(q_0, w)$ for $w \in \Sigma^n$ for a single $n$, where $n$ is large?

Example
All elements of $\Delta$ appear as values of $T(q_0, .)$ for „good“ naturally induced transducer.

Do all elements of $\Delta$ appear simultaneously as values of $T(q_0, w)$ for $w \in \Sigma^n$ for a single $n$, where $n$ is large?

Example
All elements of $\Delta$ appear as values of $T(q_0, .)$ for „good“ naturally induced transducer.
Do all elements of $\Delta$ appear simultaneously as values of $T(q_0, w)$ for $w \in \Sigma^n$ for a single $n$, where $n$ is large?

Example
All elements of $\Delta$ appear as values of $T(q_0, .)$ for „good“ naturally induced transducer.

Do all elements of $\Delta$ appear simultaneously as values of $T(q_0, w)$ for $w \in \Sigma^n$ for a single $n$, where $n$ is large?

Example

\[
\begin{array}{cccc}
\text{a} & 0,1 & \text{b} \\
\text{ } & 0,1 & \text{ } \\
\end{array}
\]

\[
\begin{array}{cccc}
a, b & 0 \mid (12) \\
\text{ } & 1 \mid (12) \\
\end{array}
\]
Do all elements of $\Delta$ appear simultaneously as values of $T(q_0, w)$ for $w \in \Sigma^n$ for a single $n$, where $n$ is large? The key point is to avoid periodic behavior.

Example

$$\begin{array}{c}
00,01,10,11 & 00,01,10,11 \\
\downarrow & \downarrow \\
a & b \\
\end{array} \quad \begin{array}{c}
00 | \text{id, 01 | id} \\
10 | \text{id, 11 | id} \\
\downarrow & \downarrow \\
a, b \\
\end{array}$$
Do all elements of $\Delta$ appear simultaneously as values of $T(q_0, w)$ for $w \in \Sigma^n$ for a single $n$, where $n$ is large? The key point is to avoid periodic behavior.

Example

```
00,01,10,11  00,01,10,11
  ▼  ▼
a    b
```

```
00 | id, 01 | id
10 | id, 11 | id
  ▼  ▼
a, b
```
Do all elements of $\Delta$ appear simultaneously as values of $T(q_0, w)$ for $w \in \Sigma^n$ for a single $n$, where $n$ is large? The key point is to avoid periodic behavior.

Example

\[
\begin{array}{c}
\text{00,01,10,11} & \text{00,01,10,11} \\
\arrows{a} & \arrows{b} \\
\text{00} | \text{id, 01} | \text{id} & \text{10} | \text{id, 11} | \text{id} \\
\arrows{a,b}
\end{array}
\]
Do all elements of $\Delta$ appear simultaneously as values of $T(q_0, w)$ for $w \in \Sigma^n$ for a single $n$, where $n$ is large? The key point is to avoid periodic behavior.

Example

\begin{align*}
00,01,10,11 & \quad 00,01,10,11 \\
\rightarrow & \quad \rightarrow \\
a & \quad b \\
\end{align*}

\begin{align*}
00 \mid \text{id, 01} \mid \text{id} & \\
10 \mid \text{id, 11} \mid \text{id} \\
\rightarrow & \quad \rightarrow \\
a, b & \quad a, b
\end{align*}
Definition (Representation)

Let $G$ be a finite group and $k \in \mathbb{N}$. A **representation** of rank $k$ is a continuous homomorphism $D : G \rightarrow \mathbb{C}^{k \times k}$.

Lemma

Let $f$ be a continuous function from $G$ to $\mathbb{C}$. There exists $r \in \mathbb{N}$ and unitary, irreducible representations $D^{(\ell)} = (d^{(\ell)}_{i,j})_{i,j < k_\ell}$ along with $c_\ell \in \mathbb{C}$ such that

$$f(g) = \sum_{\ell < r} c_\ell d^{(\ell)}_{i_\ell,j_\ell}(g)$$

holds for all $g \in G$. 
**Definition (Representation)**

Let $G$ be a finite group and $k \in \mathbb{N}$. A **Representation** of rank $k$ is a continuous homomorphism $D : G \rightarrow \mathbb{C}^{k \times k}$.

**Lemma**

Let $f$ be a continuous function from $G$ to $\mathbb{C}$. There exists $r \in \mathbb{N}$ and unitary, irreducible representations $D^{(\ell)} = (d^{(\ell)}_{i,j})_{i,j<k_\ell}$ along with $c_\ell \in \mathbb{C}$ such that

$$f(g) = \sum_{\ell<r} c_\ell d^{(\ell)}_{i_\ell,j_\ell}(g)$$

holds for all $g \in G$. 
Lemma

Suppose that

\[ \sum_{n<N} D(T(n)) \mu(n) = o(N) \]

holds for all irreducible unitary representations of \( G \). Then \( u = (u_n)_{n \geq 0} \) is orthogonal to \( \mu(n) \).
We follow the method of Mauduit and Rivat that they use for studying the Rudin-Shapiro sequence.

(Adopted) Definition

Let $U(n)$ be a sequence of unitary matrices. We say that $U$ has the **Fourier property** if there exists $\eta > 0$ and $c$ such that for all $\lambda, \alpha$ and $t$

$$\left\| \frac{1}{k^\lambda} \sum_{m<k^\lambda} U(mk^\alpha)e(mt) \right\| \leq ck^{-\eta\lambda}.$$ 

Carry Property: the contribution of high digits and the contribution of low digits are „independent“. 
We follow the method of Mauduit and Rivat that they use for studying the Rudin-Shapiro sequence.

**(Adopted) Definition**

Let $U(n)$ be a sequence of unitary matrices. We say that $U$ has the **Fourier property** if there exists $\eta > 0$ and $c$ such that for all $\lambda$, $\alpha$ and $t$

$$\left\| \frac{1}{k^\lambda} \sum_{m < k^\lambda} U(mk^\alpha) e(mt) \right\| \leq ck^{-\eta \lambda}.$$  

**Carry Property:** the contribution of high digits and the contribution of low digits are „independent“.
We follow the method of Mauduit and Rivat that they use for studying the Rudin-Shapiro sequence.

**Definition**

Let $U(n)$ be a sequence of unitary matrices. We say that $U$ has the **Fourier property** if there exists $\eta > 0$ and $c$ such that for all $\lambda, \alpha$ and $t$

$$\left\| \frac{1}{k^\lambda} \sum_{m<k^\lambda} U(mk^\alpha) e(mt) \right\| \leq ck^{-\eta \lambda}.$$

**Carry Property**: the contribution of high digits and the contribution of low digits are „independent“.
Let $D$ be a unitary and irreducible representation of $G$.

(Adopted) Theorem

Suppose that $D \circ T$ has the Fourier property. Then we have for any real $\theta$

$$\left\| \sum_{n<N} \mu(n)D(T(n))e(\theta n) \right\| \ll c_1(k)(\log N)^{c_2(k)}N^{1-\eta'}$$

(Adopted) Theorem

Suppose that $D \circ T$ has the Fourier property. Then we have for any real $\theta$

$$\left\| \sum_{n<N} \Lambda(n)D(T(n))e(\theta n) \right\| \ll c_1(k)(\log N)^{c_3(k)}N^{1-\eta'}$$
Let $D$ be a unitary and irreducible representation of $G$.

(Adopted) Theorem
Suppose that $D \circ T$ has the Fourier property. Then we have for any real $\theta$

$$\left\| \sum_{n < N} \mu(n) D(T(n)) e(\theta n) \right\| \ll c_1(k)(\log N)^{c_2(k)} N^{1-\eta'}$$

(Adopted) Theorem
Suppose that $D \circ T$ has the Fourier property. Then we have for any real $\theta$

$$\left\| \sum_{n < N} \Lambda(n) D(T(n)) e(\theta n) \right\| \ll c_1(k)(\log N)^{c_3(k)} N^{1-\eta'}$$
Ideas for the proof

Vaughan method:

Estimating

\[ S_I(\theta) = \sum_{m} \left| \sum_{\substack{n \in \mathbb{N} \cap I}} f(mn) e(\theta mn) \right| \]

\[ S_{II}(\theta) = \sum_{m} \sum_{n} a_m b_n f(mn) e(\theta mn) \]

provides estimates for

\[ \sum_{n < N} \mu(n)f(n), \quad \sum_{n < N} \Lambda(n)f(n) \]

Use variants of the Van-der-Corput inequality and the carry property to remove the contribution of low digits.

Use the Fourier property.
Ideas for the proof

Vaughan method:
Estimating

\[
S_I(\theta) = \sum_m \left| \sum_{n \mid mn \in I} f(mn) e(\theta mn) \right|
\]

\[
S_{II}(\theta) = \sum_m \sum_n a_m b_n f(mn) e(\theta mn)
\]

provides estimates for

\[
\sum_{n < N} \mu(n) f(n), \quad \sum_{n < N} \Lambda(n) f(n)
\]

Use variants of the Van-der-Corput inequality and the carry property to remove the contribution of low digits.

Use the Fourier property.
Ideas for the proof

Vaughan method:

Estimating

\[
S_I(\theta) = \sum_m \left| \sum_{mn \in I} f(mn) e(\theta mn) \right|
\]

\[
S_{II}(\theta) = \sum_m \sum_n a_m b_n f(mn) e(\theta mn)
\]

provides estimates for

\[
\sum_{n<N} \mu(n) f(n), \quad \sum_{n<N} \Lambda(n) f(n)
\]

Use variants of the Van-der-Corput inequality and the carry property to remove the contribution of low digits.

Use the Fourier property.
Problem: Distinguish representations $D$ that fulfill the Fourier Property.

**Lemma**
Let $A$ be a DFA and $\mathcal{T}_A$ a naturally induced inducer. There exists $d'$ and representations $D_0, \ldots, D_{d'-1}$ such that

$$D_\ell(T(q, (n)_k)) = e\left(\frac{n\ell}{d'}\right).$$

**Theorem**
Let $D$ be a unitary and irreducible representation different from $D_0, \ldots, D_{d'-1}$. Then $D(T(\cdot))$ has the Fourier Property.
Problem: Distinguish representations $D$ that fulfill the Fourier Property.

**Lemma**

Let $A$ be a DFA and $T_A$ a naturally induced inducer. There exists $d'$ and representations $D_0, \ldots, D_{d'-1}$ such that

$$D_\ell(T(q, (n)_k)) = e\left(\frac{n\ell}{d'}\right).$$

**Theorem**

Let $D$ be a unitary and irreducible representation different from $D_0, \ldots, D_{d'-1}$. Then $D(T(\cdot))$ has the Fourier Property.
Problem: Distinguish representations $D$ that fulfill the Fourier Property.

**Lemma**

Let $A$ be a DFA and $\mathcal{T}_A$ a naturally induced inducer. There exists $d'$ and representations $D_0, \ldots, D_{d'-1}$ such that

$$D_\ell(T(q, (n)_k)) = e\left(\frac{n\ell}{d'}\right).$$

**Theorem**

Let $D$ be a unitary and irreducible representation different from $D_0, \ldots, D_{d'-1}$. Then $D(T(\cdot))$ has the Fourier Property.
The treatment is very similar to the orthogonality to the Möbius function.

One has to work more carefully to extract the main term. The actual frequencies can be made explicit and are determined by the behavior of the automatic sequence along arithmetic progressions.
Automatic Sequences along Primes

The treatment is very similar to the orthogonality to the Möbius function.
One has to work more carefully to extract the main term.
The actual frequencies can be made explicit and are determined by the behavior of the automatic sequence along arithmetic progressions.
Automatic Sequences along Primes

The treatment is very similar to the orthogonality to the Möbius function. One has to work more carefully to extract the main term. The actual frequencies can be made explicit and are determined by the behavior of the automatic sequence along arithmetic progressions.
The treatment is very similar to the orthogonality to the Möbius function. One has to work more carefully to extract the main term. The actual frequencies can be made explicit and are determined by the behavior of the automatic sequence along arithmetic progressions.