Möbius disjointness along ergodic sequences for uniquely ergodic actions

Joanna Kułaga-Przymus

Nicolaus Copernicus University

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(joint work with M. Lemańczyk)

Outline

Möbius disjointness

- $\boldsymbol{\mu}(n) = \text{Möbius function} = \begin{cases} (-1)^k, & \text{if } n = p_1 \cdot \ldots \cdot p_k, \\ 1, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$
- Sarnak's conjecture (2011):

$$\boxed{\sum_{n\leq N} f(T^n x)\boldsymbol{\mu}(n) = \mathrm{o}(N)} \quad (S)$$

whenever $T\colon X \to X$ is a zero entropy homeomorphism of a compact metric space, $f \in C(X)$, $x \in X$.
If the above holds, we speak of the Möbius disjointness of T.

Instead of $\boldsymbol{\mu}$ one can also consider a multiplicative function $\boldsymbol{u}$ with $|\boldsymbol{u}| \leq 1$ and study an analogous condition (and speak about $\boldsymbol{u}$-disjointness).

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Möbius disjointness along random sequences

We will be interested in the following variation of (S):

$$\boxed{\sum_{n\leq N} f(R^{a_n}z)\boldsymbol{u}(n) = \mathrm{o}(N)} \quad (*)$$

for each uniquely ergodic $R\colon Z\to Z$, $f\in C(Z)$ with $\int f = 0$, $z\in Z$ and each mutliplicative $\boldsymbol{u}\colon \mathbb{N}\to\mathbb{C}$, $|\boldsymbol{u}|\leq 1$.

- Looking at sequences (a_n) allows us to look at the actions of lcsc groups, not only $G=\mathbb{Z}$, e.g. at flows.
- In fact, we will produce "good" $(a_n)\subset\mathbb{Z}$ satisfying $(*)$ by producing "good" $(b_n)\subset\mathbb{R}$, taking $a_n := [b_n]$ and using suspension flows.
- The existence of $(a_n)\subset\mathbb{Z}$ for which $(*)$ holds for $\boldsymbol{u}=\boldsymbol{\mu}$ is not surprising: for any $\boldsymbol{u}$ with $\sum_{n\leq N}\boldsymbol{u}(n)=\mathrm{o}(N)$, it suffices to assume that (a_n) is increasing sufficiently slowly.
  E.g. for $\boldsymbol{u}=\boldsymbol{\mu}$, we can take $([n^c])$ with $0<c<1$.[1]

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Our sequences **differ from the slowly increasing sequences.**

Let $F\colon \mathbb{N} \to \mathbb{C}$ be bdd. Suppose that $\sum_{n\leq N} F(rn)\overline{F(sn)} = \mathrm{o}(N)$ for any sufficiently large primes $r \neq s$. Then

$$\sum_{n\leq N} F(n)\boldsymbol{u}(n) = \mathrm{o}(N)$$

for any multiplicative function $\boldsymbol{u}$ with $|\boldsymbol{u}| \leq 1$.[2]

$F(n) = f(T^n x)$ for $n \in \mathbb{Z}, x \in X$ and $f \in C(X)$.

- Our sequences $(b_n) \subset \mathbb{R}$ will satisfy $\sum_{n\leq N} f(S_{b_{pn}}x)\overline{f(S_{b_{qn}}x)} = \mathrm{o}(N)$ for any $f \in C(X)$, $\int f = 0$ and **any uniquely ergodic flow** $\mathcal{S} = (S_t)_{t\in\mathbb{R}}$. In particular, $\sum_{n\leq N} f(S_{b_n}x)\boldsymbol{u}(n) = \mathrm{o}(N)$.[3]
- Note that this will fail for a slowly increasing sequence $(a_n) \subset \mathbb{Z}$ whenever $\sum_{n\leq N} \boldsymbol{u}(n) \neq \mathrm{o}(N)$.

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[3] For automorphisms we consider $a_n = [b_n]$ and have analogous propertie for each uniquely ergodic R.

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Our sequences also **differ from $a_n = n$**.

- $(-1)^{n+1}$ is multiplicative and at the same time this sequence is uniquely ergodic. $(-1)^{n+1}$ is not orthogonal to itself!
- We do not expect Möbius disjointness ($\boldsymbol{u} = \boldsymbol{\mu}$) along $a_n = n$ to hold for all uniquely ergodic automorphisms.[4]
  This would imply that the Chowla conjecture fails:
  - If Chowla conjecture holds then in particular μ is generic for an ergodic measure.
  - Any sequence generic for an ergodic measure can be approximated by a sequence generic for a uniquely ergodic dynamical system.[5]

Our sequences won't be increasing. They will origin from random constructions (and depend on an additional parameter).

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Why random sequences?

Recall that in the classical setting we study the convergence of

$$\frac{1}{N}\sum_{n\leq N} f(S^n y)\boldsymbol{\mu}(n).$$

We will have "random homeomorphisms" $\mathcal{S} = (S_x)_{x\in X}$ on Y and study

$$\frac{1}{N}\sum_{n\leq N} f(S_x^{(n)}(y))\boldsymbol{\mu}(n) \qquad (*).$$

Instead of S^n, we will deal with "random powers" $(S_x^{(n)})$, i.e. there will be a homeomorphism $T\colon X \to X$ and

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Goal: find T and $x \mapsto S_x$ such that $(*)$ holds for "all uniquely ergodic $\mathcal{S}$", all x, y and all $f \in C(Y)$.

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Basic notions: factors, extensions and joinings

Let $T \in Aut(X, \mathcal{B}, \mu)$, $S \in Aut(Y, \mathcal{C}, \nu)$.

1. S is called a factor of T if $S \circ \pi = \pi \circ T$ and $\pi_*(\mu) = \nu$ for some $\pi\colon X \to Y$. We also say that T is an extension of S and write $T\colon \mathcal{B} \to \mathcal{C}$ or $T \to S$.
2. Measure λ on $(X \times Y, \mathcal{B} \otimes \mathcal{C})$ is called a joining of T and S if
   - $(T \times S)_*(\lambda) = \lambda$,
   - $\lambda|_{\mathcal{B} \otimes \{\emptyset, Y\}} = \mu$ and $\lambda|_{\{\emptyset, X\} \otimes \mathcal{C}} = \nu$.

   Notation: $J(T, S)$, $J^e(T, S)$.

   - Both T and S are factors of each of their joinings.
   - $J^e(T, S)$ is non-empty iff T and S are ergodic.
   - If $\mu \otimes \nu$ is the only element of $J(T, S)$, we say that T and S are disjoint.[6]
     E.g. Id is disjoint from T, whenever T is ergodic.

[6] Note that this implies that at least one of them must be ergodic.

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Let $T \in Aut(X, \mathcal{B}, \mu)$, $S \in Aut(Y, \mathcal{C}, \nu)$.

1. S is called a factor of T if $S \circ \pi = \pi \circ T$ and $\pi_*(\mu) = \nu$ for some $\pi \colon X \to Y$. We also say that T is an extension of S and write $T \colon \mathcal{B} \to \mathcal{C}$ or $T \to S$.
2. Measure λ on $(X \times Y, \mathcal{B} \otimes \mathcal{C})$ is called a joining of T and S if
   - $(T \times S)_*(\lambda) = \lambda$,
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   Notation: $J(T, S)$, $J^e(T, S)$.

   - Both T and S are factors of each of their joinings.
   - $J^e(T, S)$ is non-empty iff T and S are ergodic.
   - If $\mu \otimes \nu$ is the only element of $J(T, S)$, we say that T and S are disjoint.[6]

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Tools to prove Möbius disjointness and beyond

Let $T \in Aut(X, \mathcal{B}, \mu)$ be ergodic.

- AOP (asymptotical orthogonality of powers)[7] is one of the most useful tools to prove Möbius disjointness for a particular dynamical system. We say that T has AOP whenever

$$\limsup_{p \neq q, p,q \to \infty} \sup_{\kappa \in J^e(T^p, T^q)} \left| \int_{X \times X} f \otimes g \, d\kappa \right| = 0.$$

- If T satisfies AOP, then Möbius disjointness holds in any uniquely ergodic model of T.[7]
  - Recall that (S, Y) is a uniquely ergodic model of $(X, \mathcal{B}, \mu, T)$ if $(X, \mathcal{B}, \mu, T) \simeq (Y, \mathcal{B}(Y), \nu, S)$, where $\mathcal{B}(Y)$ is the sigma-algebra of Borel sets and ν is the unique S-invariant measure.
  - Each ergodic T has a uniquely ergodic model.[8]

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Assume that T is a uniquely ergodic homeomorphism of X.

- AOP implies strong MOMO (Möbius orthogonality of moving orbits) [relative to $\boldsymbol{u}$]:[9]

$$\lim_{K\to\infty} \frac{1}{b_{K+1}} \sum_{k\leq K} \left| \sum_{b_k\leq n<b_{k+1}} f(T^n x_k)\boldsymbol{u}(n) \right| = 0,$$

  for each multiplicative $\boldsymbol{u}$, $|\boldsymbol{u}| \leq 1$, each $(b_k) \subset \mathbb{N}$ with $b_{k+1} - b_k \to \infty$ and each choice of $x_k \in X$, $k \geq 1$, and $f \in C(X)$, $\int f\, d\mu = 0$.

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$$\frac{1}{M} \sum_{M\leq m<2M} \left| \frac{1}{H} \sum_{m\leq h<m+H} f(T^h x)\boldsymbol{u}(h) \right| \to 0$$

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1. Compact group extensions:
   G – compact group, $\varphi\colon X \to G$ – measurable

$$T_\varphi(x,g) = (Tx, \varphi(x)g)$$

   (the same formula for lscs groups)

2. Isometric extensions:
   all "intermediate" extensions of cpt. group extensions, i.e. $T:\mathcal{B}\to\mathcal{A}$ is isometric if we have $\mathcal{C}\supset\mathcal{B}\supset\mathcal{A}$ and $\overline{T}:\mathcal{C}\to\mathcal{A}$ is compact group extension.

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is ergodic for $T \times T$.

- E.g.: $S \times T \to S$ is relatively WM $\iff$ T is WM
- Relatively weakly mixing extensions are relatively disjoint from relatively distal extensions.
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While there were some results concerning lifting Möbius disjointness to distal extensions of rotations:

- Green, Tao 2012: affine unipotent diffeomorphisms on G/Γ;
- Liu, Sarnak 2015: analytic Anzai skew products (+ an extra assumption) over rotations, all zero entropy affine systems;
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Basic notions: Rokhlin extensions

Assume that:

- $T \in Aut(X, \mathcal{B}, \mu)$ is ergodic,
- $\varphi \colon X \to G$ is a cocycle with values in a lcsc Abelian group,
- $G \ni g \mapsto S_g \in Aut(Y, \mathcal{C}, \nu)$ is a (measurable) G-representation (denoted by $\mathcal{S}$) that is ergodic.

Then

$$T_{\varphi,\mathcal{S}}(x, y) = (Tx, S_{\varphi(x)}(y)) \text{ for } (x, y) \in X \times Y$$

is called a Rokhlin extension of T.

In particular, we can take $G = \mathbb{R}$ and $\mathcal{S}$=a flow.

- E.g. if $\varphi = const = g$ then $T_{\varphi,\mathcal{S}}(x, y) = (Tx, S_g y)$, i.e. we obtain the direct product $T \times S_g$.

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Properties of $T_{\varphi,\mathcal{S}}$:[12]

- $T_{\varphi,\mathcal{S}}$ is ergodic $\iff$ T is ergodic and $\sigma_{\mathcal{S}}(\Lambda_\varphi) = 0$.
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- If T is uniquely ergodic, φ is ergodic and $\mathcal{S}$ is uniquely ergodic then $T_{\varphi,\mathcal{S}}$ is uniquely ergodic.
- If φ is ergodic and $\mathcal{S}$ is weakly mixing then $T_{\varphi,\mathcal{S}} \to T$ is relatively weakly mixing.

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Basic notions: cocycles

Let $T \in Aut(X, \mathcal{B}, \mu)$, let $\varphi \colon X \to G$ be measurable, with values in a lcsc Abelian group. Consider the group extension:

$$T_\varphi(x, g) = (Tx, \varphi(x) + g) \text{ for } x \in X, g \in G.$$

Note that $(T_\varphi)^k(x, g) = (T^k x, \varphi^{(k)}(x) + g)$, where

$$\varphi^{(k)}(x) = \begin{cases} \varphi(x) + \varphi(Tx) + \ldots + \varphi(T^{k-1}x) & \text{if} \quad k \geq 1, \\ 0 & \text{if} \quad k = 0, \\ -(\varphi(T^k x) + \ldots + \varphi(T^{-1}x)) & \text{if} \quad k < 0. \end{cases}$$

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$$\boxed{T_\varphi(x,g) = (Tx, \varphi(x) + g) \text{ for } x \in X, g \in G.}$$

Note that $(T_\varphi)^k(x,g) = (T^k x, \varphi^{(k)}(x) + g)$, where

$$\varphi^{(k)}(x) = \begin{cases} \varphi(x) + \varphi(Tx) + \ldots + \varphi(T^{k-1}x) & \text{if} \quad k \geq 1, \\ 0 & \text{if} \quad k = 0, \\ -(\varphi(T^k x) + \ldots + \varphi(T^{-1}x)) & \text{if} \quad k < 0. \end{cases}$$

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Let $\tau = (\tau_g)_{g\in G}$ be the natural G-action on $(X \times G, \mu \otimes \lambda_G)$:

$$\tau_g(x,g') = (x, g+g') \text{ for } (x,g') \in X \times G.$$

Then τ preserves $\mu \otimes \lambda_G$. Let $\lambda \simeq \lambda_G$ be a probability measure.
τ with respect to $\mu \otimes \lambda$ is non-singular.
Notice that $T_\varphi \circ \tau_g = \tau_g \circ T_\varphi$ for $g \in G$.
Thus, τ acts on the σ-algebra of T_φ-invariant sets.
Notation: $\mathcal{W}(\varphi)$, $\mathcal{W}(\varphi, T, \mu)$.
It is a non-singular ergodic action (with respect to $\lambda \simeq \lambda_G$).

Basic notions: Mackey action

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- Suppose that φ is recurrent (i.e. $\varphi^{(n)}(x)$ visits each neighborhood of $0 \in G$ infinitely often for a.e. x). Then $h(T_{\varphi,\mathcal{S}}) = h(T)$ for each $\mathcal{S}$.[13]
- If φ is ergodic then it is recurrent.
- In particular, if $h(T) = 0$ and φ is ergodic, $h(T_{\varphi,\mathcal{S}}) = 0$ for each $\mathcal{S}$.

[13] Danilenko 2001

Basic notions: Rokhlin extensions and entropy

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Outline

AOP for Rokhlin extensions

Theorem (KP, Lemańczyk)

Assume that T has the AOP property and for each $r \neq s$, $r, s \in \mathcal{P}$ and arbitrary $\eta \in J^e(T^r, T^s)$:

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- *the Mackey action $\mathcal{W}((\varphi^{(r)} \times \varphi^{(s)}, T^r \times T^s, \eta)$ is weakly mixing.*

Let $\mathcal{S} = (S_g)_{g \in G}$ be an ergodic G-action on $(Y, \mathcal{C}, \nu)$. Then $T_{\varphi,\mathcal{S}}$ has the AOP property.

- $((T_\varphi)^r \times T^s)(x, t, y) = (T^r x, \varphi^{(r)}(x) + t, T^s y) = (T^r \times T^s)_{\varphi^{(r)}}(x, y, t)$
- A G-action $\mathcal{W}$ is weakly mixing $\iff$ the only L^∞ eigenvalue of $\mathcal{W}$ is the trivial character.
- $\chi \in \widehat{G}$ is an L^∞-eigenvalue if for some $0 \neq f \in L^\infty$, we have $f \circ W_g = \chi(g) \cdot f$ for all $g \in G$.

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Ergodic sequences

Recall that $(a_n) \subset G$ is called ergodic if for each ergodic $\mathcal{S} = (S_g)_{g \in G} \subset Aut(Y, \mathcal{C}, \nu)$, we have

$$\boxed{\frac{1}{N} \sum_{n \le N} f \circ S_{b_n} \to \int f \, d\nu \text{ in } L^2}$$

for each $f \in L^2(Y, \mathcal{C}, \nu)$.

- Let T be uniquely ergodic, let $\varphi \colon X \to G$ be continuous and let $\mathcal{W}(\varphi)$ be weakly mixing.
  Then $(\varphi^{(n)}(x))$ is ergodic for each $x \in X$. In particular, the assertion holds if φ is ergodic.[14]

In our setting $(T_\varphi)^r \times (T_\varphi)^s$ is ergodic $\Rightarrow (T_\varphi)^r$ is ergodic $\Rightarrow T_\varphi$ is ergodic $\Rightarrow (\varphi^{(n)}(x))$ is ergodic.

- $([n^c])$, $c \in (0, 1)$, is also ergodic.[15]

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Suppose now that T is a totally ergodic rotation.

- Then T has the AOP property.[16]
- WLOG: X is a compact monothetic group, $Tx = x + \alpha$, where $\{n\alpha : n \in \mathbb{Z}\}$ is dense in X.

We describe now $J^e(T^r, T^s)$. Let $a, b \in \mathbb{Z}$ so that $ar + bs = 1$. Fix $u \in X$ and consider $A_u := \{(x, y + u) \in X \times X : sx = ry\}$.

V_u is given by $V_u(x, y + u) = ax + by$. Since T is uniquely ergodic, each ergodic joining of T^r and T^s is supported on one of the sets A_u.

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AOP for relatively weakly mixing extensions of $Tx = x + \alpha$

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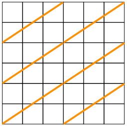

Figure: A_0 for $s = 3$, $r = 2$

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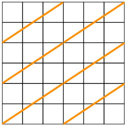

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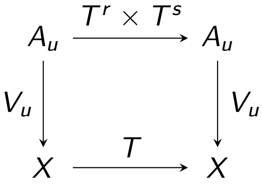

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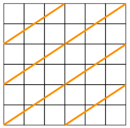

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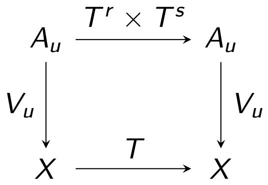

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Theorem (KP, Lemańczyk)

Assume that $f \in C^{1+\delta}(\mathbb{T})$, $\delta > 0$, $\int_{\mathbb{T}} f \, d\lambda_{\mathbb{T}} = 0$, not a trigonometric polynomial. Then, for a generic α, for $Tx = x + \alpha$:

- $f^{(r)}(r\cdot)$ *is ergodic for each* $r \in \mathcal{P}$;
- $\mathcal{W}(f^{(r)}(r\cdot) \times f^{(s)}(s \cdot +u))$ *is weakly mixing for* $r < s$ *in* $\mathcal{P}$.

In particular, for each ergodic flow $\mathcal{S} = (S_t)_{t \in \mathbb{R}} \subset Aut(Y, \mathcal{C}, \nu)$, $T_{\varphi,\mathcal{S}}$ has the AOP property.

Rokhlin extensions with AOP – consequences

Suppose that T is uniquely ergodic $\varphi\colon X \to \mathbb{R}$ is continuous and ergodic, $\mathcal{S} = (S_t)_{t\in\mathbb{R}}$ is uniquely ergodic and we have AOP for $T_{\varphi,\mathcal{S}}$. Take $F(x,y) = f(y)$. Then, by strong MOMO,

$$0 = \lim_{K\to\infty} \frac{1}{b_{K+1}} \sum_{k\le K} \left| \sum_{b_k \le n < b_{k+1}} F((T_{\varphi,\mathcal{S}})^n(x,y_k))\boldsymbol{u}(n) \right|$$

$$= \lim_{K\to\infty} \frac{1}{b_{K+1}} \sum_{k\le K} \left| \sum_{b_k \le n < b_{k+1}} f(S_{\varphi^{(n)}(x)}(y_k))\boldsymbol{u}(n) \right|$$

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where $a_n = \varphi^{(n)}(x)$ (this sequence does not depend on $\mathcal{S}$).

- If for proving AOP we use the results from the previous part of the talk, we can take **ANY uniquely ergodic** $\mathcal{S}$.

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where $a_n=\varphi^{(n)}(x)$ (this sequence does not depend on $\mathcal{S}$).

- If for proving AOP we use the results from the previous part of the talk, we can take **ANY uniquely ergodic** $\mathcal{S}$.

Rokhlin extensions with AOP – consequences

Suppose that T is uniquely ergodic $\varphi\colon X \to \mathbb{R}$ is continuous and ergodic, $\mathcal{S} = (S_t)_{t\in\mathbb{R}}$ is uniquely ergodic and we have AOP for $T_{\varphi,\mathcal{S}}$. Take $F(x,y) = f(y)$. Then, by strong MOMO,

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Suppose that T is uniquely ergodic $\varphi\colon X \to \mathbb{R}$ is continuous and ergodic, R is uniquely ergodic and we have AOP for $T_{\varphi,\widetilde{R}}$, where $\widetilde{R}$ is the suspension flow over R. Then

$$\lim_{K\to\infty} \frac{1}{b_{K+1}} \sum_{k\le K} \left| \sum_{b_k\le n<b_{k+1}} f(R^{[a_n]}(z_k))\boldsymbol{u}(n) \right| = 0,$$

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E.g. for R on $\mathbb{Z}/2\mathbb{Z}$ given by $Ri = i + 1$ we get

$$\lim_{K\to\infty} \frac{1}{b_{K+1}} \sum_{k\leq K} \left| \sum_{b_k\leq n<b_{k+1}} (-1)^{[a_n]} \boldsymbol{u}(n) \right| = 0.$$

Equivalently, as $H \to \infty$, $H/M \to 0$,

$$\frac{1}{M} \sum_{M\leq m<2M} \left| \frac{1}{H} \sum_{m\leq h<m+H} (-1)^{[a_h]} \boldsymbol{u}(h) \right| \to 0 \qquad (*)$$

Notice that the above holds **without any assumptions** on the convergence of $\frac{1}{N}\sum_{n\leq N} \boldsymbol{u}(n)$.

If $\boldsymbol{u}$ satisfies a certain condition stronger than aperiodicity[17] then $(*)$ holds for the constant sequence (a_n).[18]

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Rokhlin extensions with AOP – consequences

This theory can be applied to the affine cocycle $\varphi(x) = x - 1/2$ over $Tx = x + \alpha$.

- To make φ continuous, we use the coordinates given by the corresponding Sturmian model.

If α is irrational with bounded partial quotients and $\alpha, \beta, 1$ are rationally independent then we can take

$$a_n = \left[n\beta + \frac{n(n-1)}{2}\alpha - \frac{n}{2} - \sum_{j=1}^{n-1}[\beta + j\alpha] \right], n \geq 1.$$

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Thank you!