Mathematics of Imaging
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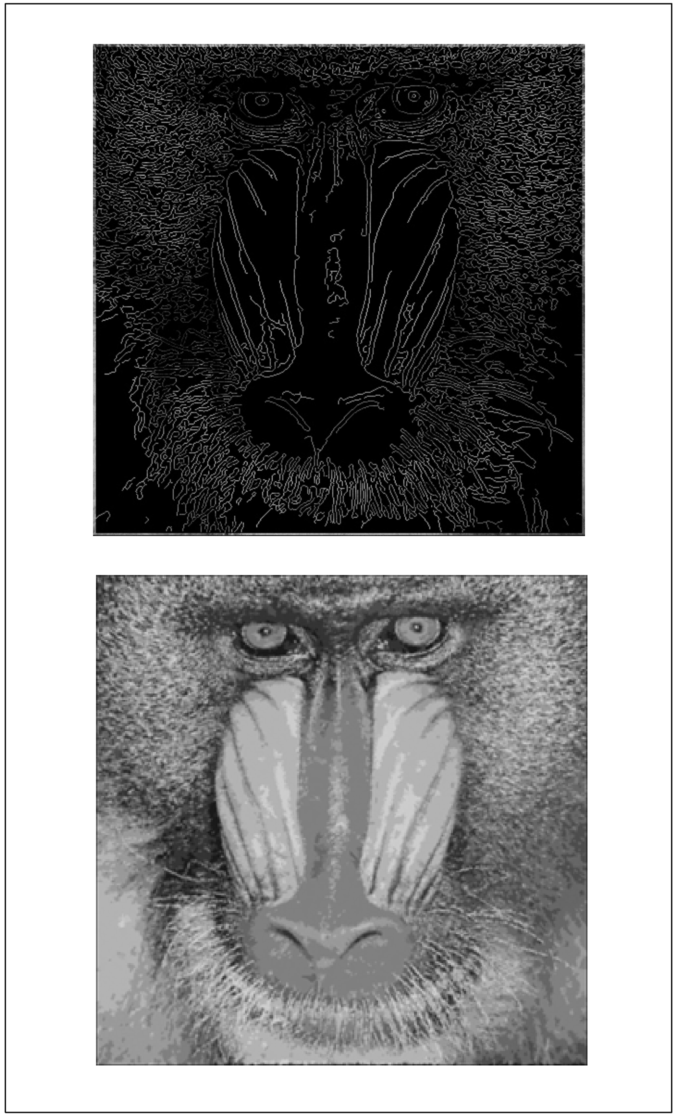

On Hausdorff Stable Image Approximation and Inpainting Method

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Compensated Convex Transforms

- For a function $f : \mathbb{R}^n \to \mathbb{R}$ of at most quadratic growth and constant $\lambda > 0$, define λ-parametrised convexity-based transforms

  (i) Lower compensated convex transform

$$C^l_\lambda(f)(x) := \text{co}\left[\lambda|\cdot|^2 + f(\cdot)\right](x) - \lambda|x|^2, \quad x \in \mathbb{R}^n$$

  (ii) Upper compensated convex transform

$$C^u_\lambda(f)(x) := \lambda|x|^2 - \text{co}\left[\lambda|\cdot|^2 - f(\cdot)\right](x), \quad x \in \mathbb{R}^n$$

  where co$[g]$ $=$ convex envelope of g, and $|\cdot| =$ Euclidean norm

- introduced in Kewei Zhang, Compensated convexity and its applications, Ann. Inst. Henri Poincaré - Analyse Nonlinéaire 25 (2008), 743-771

The Average Approximations

- Basic assumptions

  $K \subset \mathbb{R}^n$, either K compact or $K = \mathbb{R}^n \setminus \Omega$ with $\Omega \subset \mathbb{R}^n$ bounded open.

  Let $f : \mathbb{R}^n \to \mathbb{R}$ be bounded and uniformly continuous,

  define $f_K : K \to \mathbb{R}$ as $f_K(x) = f(x)$ on K and for $M > \sup_K |f(x)|$,

$$f_K^{-M}(x) = \begin{cases} f(x), & x \in K, \\ -M, & x \in \mathbb{R}^n \setminus K, \end{cases} \quad f_K^{M}(x) = \begin{cases} f(x), & x \in K, \\ M, & x \in \mathbb{R}^n \setminus K, \end{cases}$$

- The average approximation with scale $\lambda > 0$

$$A_\lambda^M(f_K)(x) = \frac{1}{2}\left(C_\lambda^l(f_K^M)(x) + C_\lambda^u(f_K^{-M})(x)\right).$$

- Remark The average approximation is a Lipschitz continuous approximation of f respectively in co(K) - the convex hull of K.

Property: Hausdorff stability

- **Theorem** Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is bounded and uniformly continuous satisfying $|f(x)| \leq A_0$ for all $x \in \mathbb{R}^n$. Assume $K,\ L \subset \mathbb{R}^n$ to be closed sets. Let $M > A_0$ and $\lambda > 0$. Then for every $\epsilon > 0$, there is a $\delta > 0$ such that for all $x \in \mathbb{R}^n$ there holds

$$|A_\lambda^M(f_K)(x) - A_\lambda^M(f_L)(x)| < \epsilon$$

whenever $\mathrm{dist}_{\mathcal{H}}(K, L) < \delta$, where

$$\mathrm{dist}_{\mathcal{H}}(K, L) := \max\left\{ \sup_{x \in K} \mathrm{dist}(x, L),\ \sup_{x \in L} \mathrm{dist}(x, K) \right\}$$

is the Hausdorff distance between K and L.

Property: Error estimates

- Let $K_\Omega = \Omega^c$ with $\Omega \subset \mathbb{R}^n$ bounded open. For $x \in \Omega$, let $d(x)$ be the length of the 'shortest bridge' through x to K_Ω and let $d_\Omega = \sup_{x\in\Omega} d(x)$.

- modulus of continuity There is a continuous, non-decreasing, concave majorant $\omega(\cdot)$ of the modulus of continuity of f such that $\omega(0) = 0$ and $\omega(t) \le a_f t + b_f$ where $a_f \ge 0$, $b_f \ge 0$.

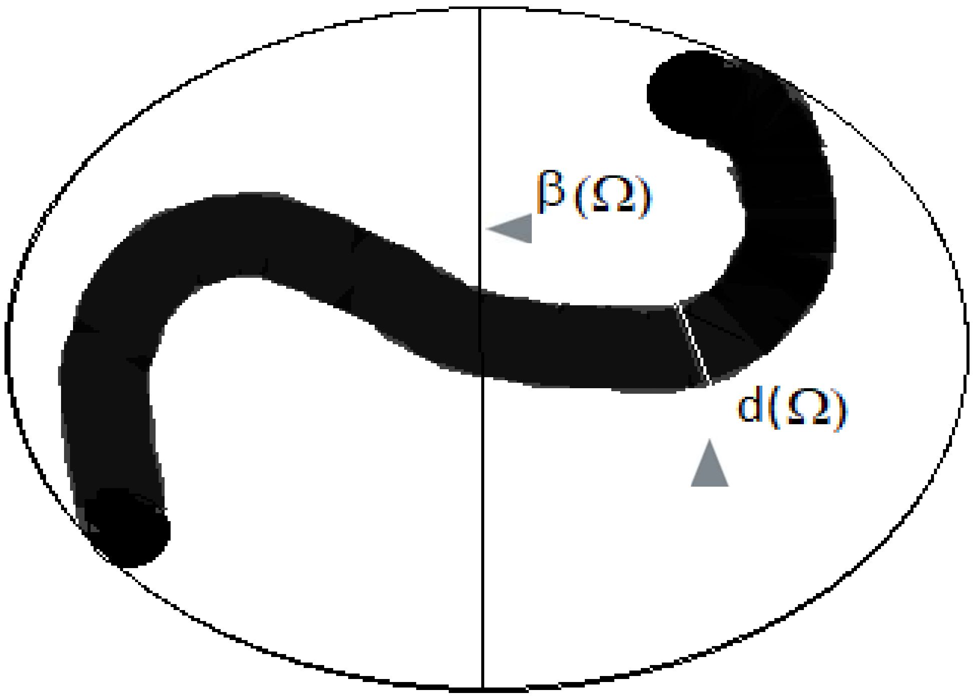
  Theorem Assume $|f| \le A_0$ and $M > A_0 + \lambda \mathrm{diam}^2_\Omega$. Let $x \in \Omega$. If f is $\omega(\cdot)$-continuous, then

$$|A_\lambda^M(f_{K_\Omega})(x) - f(x)| \le \omega\left(d(x) + \frac{a_f}{\lambda} + \sqrt{\frac{b_f}{\lambda}}\right).$$

  If f is L-Lipschitz continuous, then

$$|A_\lambda^M(f_{K_\Omega})(x) - f(x)| \le Ld(x) + \frac{L^2}{2\lambda}.$$

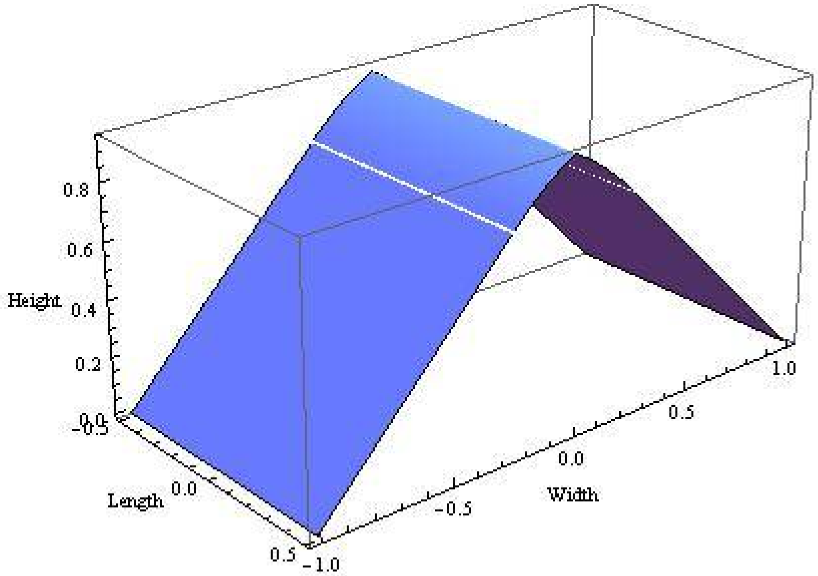
Error estimate illustration



$\beta(\Omega)$ is the minor diameter of the ellipse in Chan and Kang's inpainting error estimate,

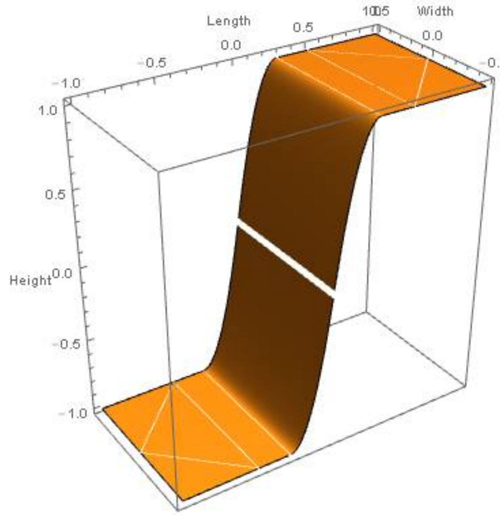
$d(\Omega)$ is the maximum width in the average inpainting error estimate

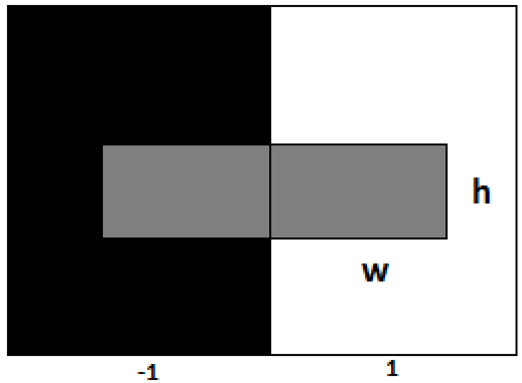
Prototype Examples



$K=\{(x,y):\ |x|=r,\ |y|\leq h\}$

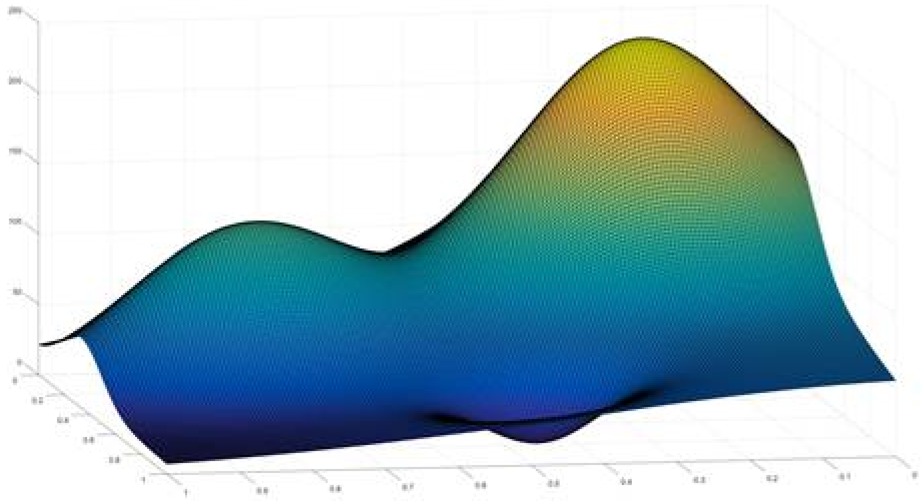
and approximation of $f_K(\pm r,y) = h-|y|$



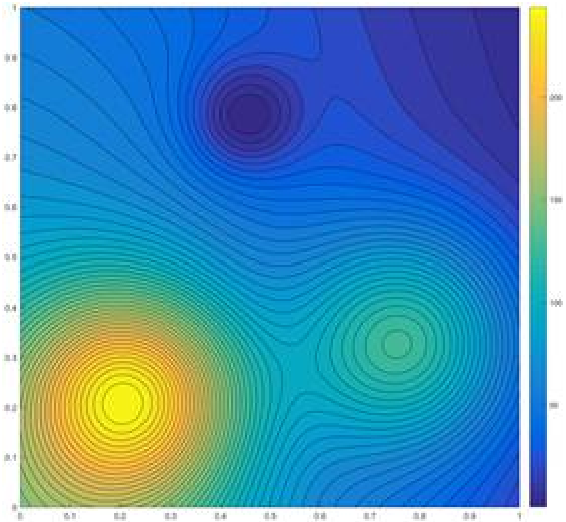


Inpainting/approximation of
a jump function with h<w.

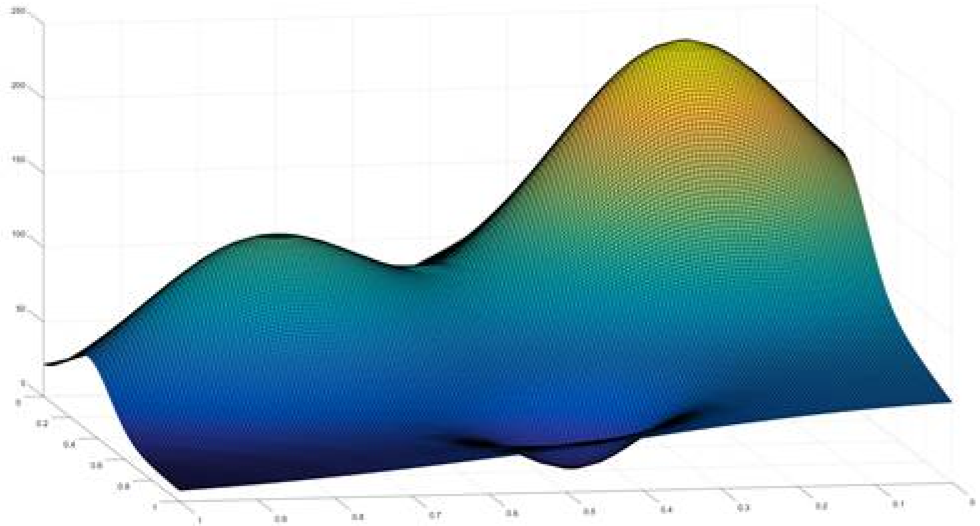
Level set Interpolation/Approximation – Franke test function



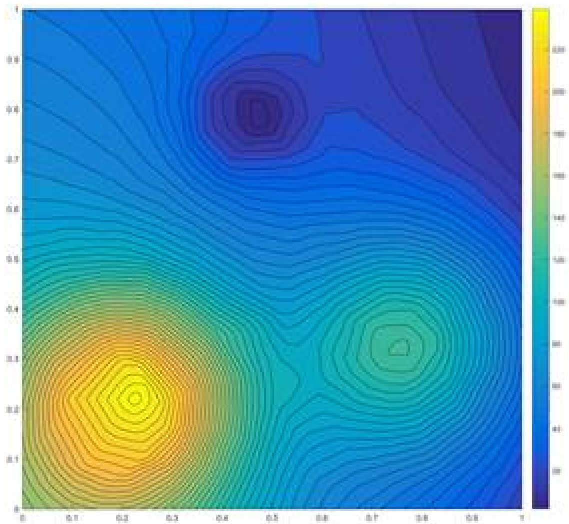
Graph of the Franke test function



Contour lines of the Franke test function



Graph of the interpolation function for the set K of 50-contour lines equally spaced

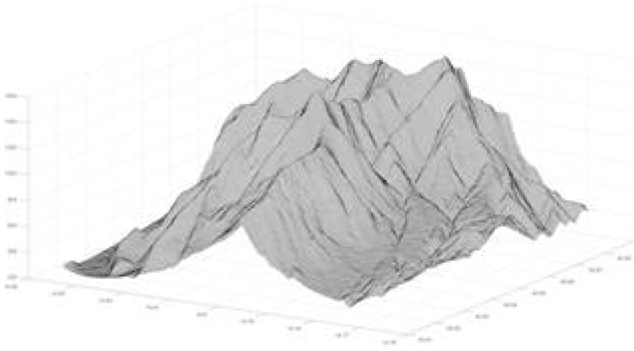


Contour lines of the Reconstructed Surface

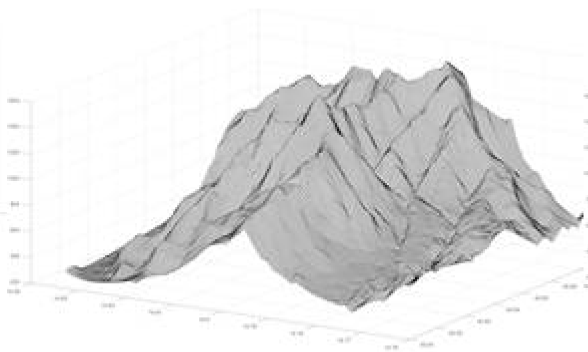
Level set Reconstruction with 10 level sets

Original Image Sample Set Image Reconstruction

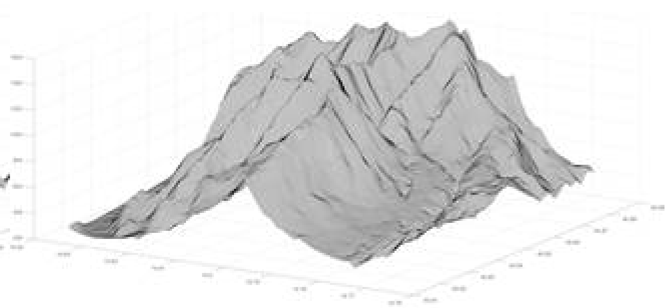
Reconstruction of Real-World Digital Elevation Maps



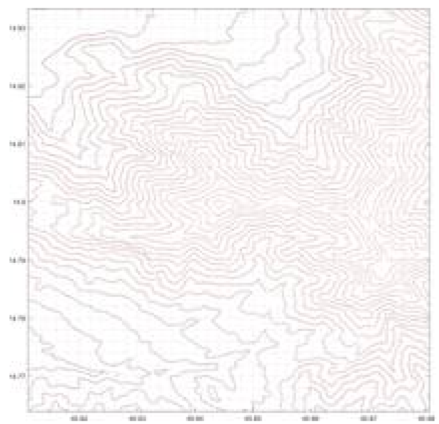
Ground truth model from USGS-STRM1 data



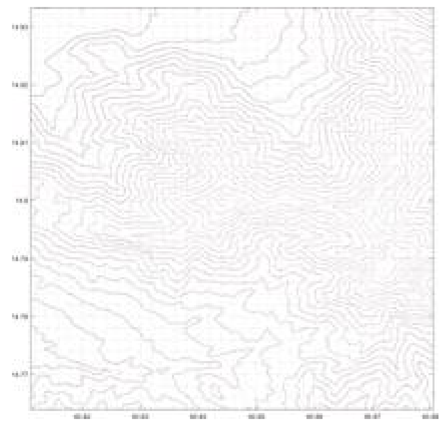
Surface reconstruction from Sample Set K_1



Surface reconstruction from Sample Set K_2

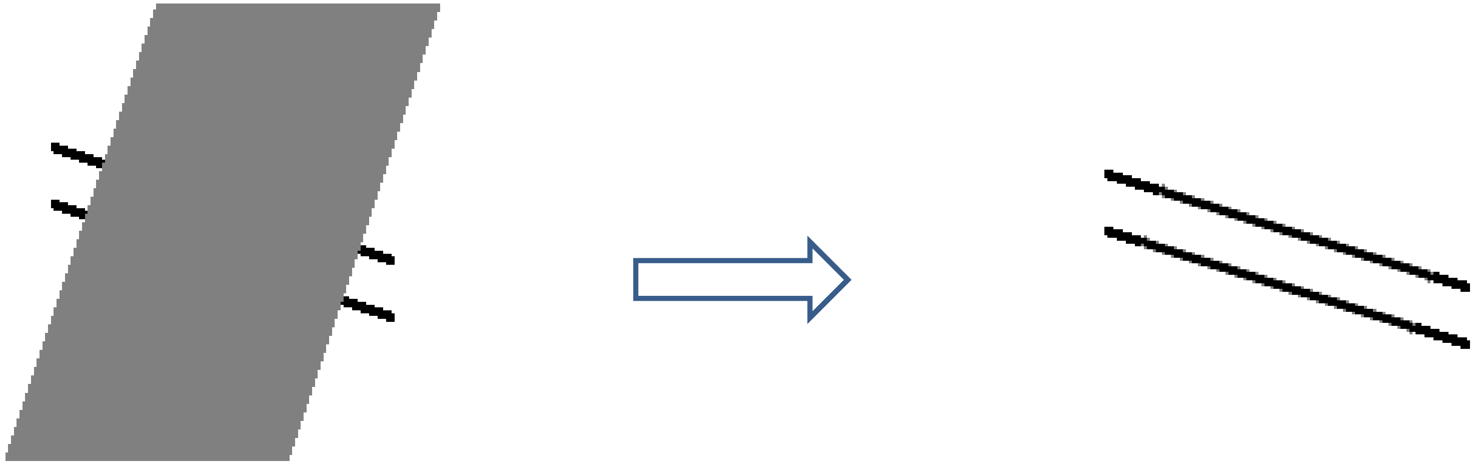


Sample Set K_1: formed by only level lines at regular height interval of 66m



Sample Set K_2: formed by taking randomly 30% of the points belonging to the level lines of the set K1 and scattered points corresponding to 5% density.

Connectivity for Image Inpainting and Salt & Pepper Noise Reconstruction for Color Images



Original Image 1920 x 1200

Reconstruction from
99.9% Salt & Pepper Noise
PSNR 20.8

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