Flash presentation : Determinantal point processes and images

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CIRM : Mathematics of Imaging

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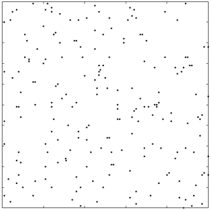
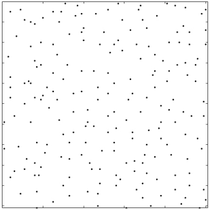
Determinantal Point Processes (DPP) model negative correlations, **repulsion** between points : the probability of observing two points close to each other is lower than in the case of the Poisson process.



(a) Sample from a Poisson point process



(b) Sample from a Determinantal point process

Definition

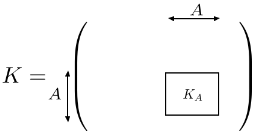
We consider $\mathcal{Y} = \{1, \ldots, N\}$ and K a hermitian matrix of size $N \times N$ such that

$$0 \preceq K \preceq 1,$$

then the random set $X \subset \mathcal{Y}$ defined by

$$\forall A \subset \mathcal{Y}, \quad \mathbb{P}(A \subset X) = \det(K_A)$$

is a determinantal point process with kernel K.



Framework of images : the data set becomes a 2D grid, $\Omega = [0, M-1] \times [0, N-1]$ and the process becomes stationary and periodic, with kernel K, a matrix of size $MN \times MN$.

Stationarity $\Rightarrow$ K becomes bloc-circulant :

$$\text{We define } C : \Omega \to \mathbb{C}, \quad K_{xy} = C(x-y).$$

$\Rightarrow$ K can be diagonalized by the Fourier matrix and its eigenvalues are the Fourier coefficients of C.

Determinantal pixel processes

Definition

Let $C : \Omega \to \mathbb{C}$ be a function defined on Ω, such that

$$\forall \xi \in \Omega, \;\; 0 \leq \widehat{C}(\xi) \leq 1.$$

Then a random subsample $X \subset \Omega$ is called a determinantal pixel process (DPixP) with kernel C, if

$$\forall A \subset \Omega, \;\; \mathbb{P}(A \subset X) = \det(K_A),$$

where K_A is the $|A| \times |A|$-matrix, $K_A = (C(x_i - x_j))_{x_i, x_j \in A}$.

Result 1 :

We can't impose a minimal distance between the points of a DPixP as total repulsion of a pair of points implies necessarily the repulsion for the whole line passing through the points.

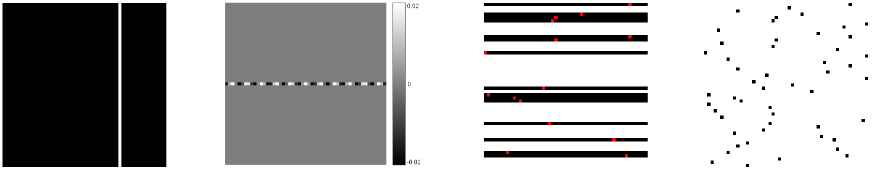


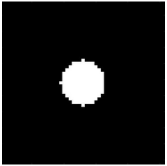
Figure: Example of a kernel inducing a hardcore repulsion in the horizontal direction. From left to right : the Fourier coefficients of C, the real part of the kernel C, a capture of the conditional density during the sampling, the final sample.

Shot noise and extreme cases of repulsion

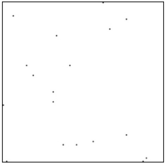
Shot noise model based on a DPixP

We consider $X \sim \text{PPixD}(C)$ and g a positive function on Ω. Then the shot noise S based on X and the spot g is defined $\forall x \in \Omega$ by

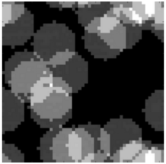
$$S(x) = \sum_{x_i \in X} g(x - x_i).$$



(a) Spot g



(b) Sample of a point process

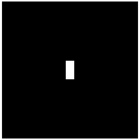


(c) Image of the related shot noise

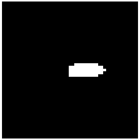
Shot noise and extreme cases of repulsion

Result 2 :

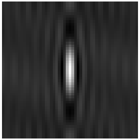
Given a fix spot function for the shot noise, we can characterize the maximal and minimal repulsion cases of these determinantal pixel processes.



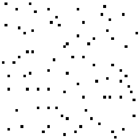
(a) Spot g



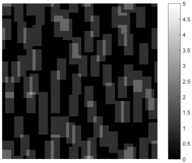
(b) Obtained Fourier coefficients



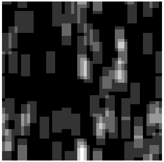
(c) Kernel C



(d) A sample of this DPixP

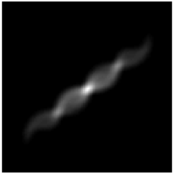


(a) Shot noise of maximal repulsion (DPixP(C))

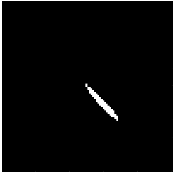


(b) Shot noise of minimal repulsion (BPP)

Shot noise and extreme cases of repulsion

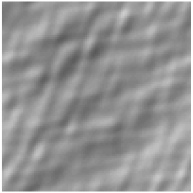


(a) Spot g



(b) Obtained Fourier coefficients

(c) Kernel C



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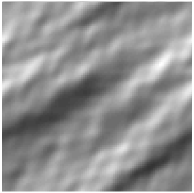

(b) Shot noise of minimal repulsion (BPP)