Lattices in Lie groups 4: Rigidity

Dave Witte Morris
University of Lethbridge, Alberta, Canada
http://people.uleth.ca/~dave.morris
Dave.Morris@uleth.ca

- Margulis Superrigidity Theorem.
- Mostow Rigidity Theorem.
- Margulis Arithmeticity Theorem.

Observation
Group homomorphism $\phi\colon \mathbb{Z}^k \to \mathbb{R}^d$
$\Rightarrow$ *ϕ extends to a continuous homo* $\hat\phi\colon \mathbb{R}^k \to \mathbb{R}^d$.

Proof.
Standard basis $\{e_1, \ldots, e_k\}$ of $\mathbb{R}^k$.
Define $\hat\phi(x_1, \ldots, x_k) = x_1\,\phi(e_1) + \cdots + x_k\,\phi(e_k)$.
("linear trans can do anything to a basis")
Linear transformation
$\Rightarrow$ homomorphism of additive groups □

Obs. Group homomorphism $\phi\colon \mathbb{Z}^k \to \mathbb{R}^d$
$\Rightarrow$ ϕ extends to a continuous homo $\hat\phi\colon \mathbb{R}^k \to \mathbb{R}^d$.

Want. Group homomorphism $\phi\colon \Gamma \to G'$
$\Rightarrow$ ϕ "extends" to a continuous homo $\hat\phi\colon G \to G'$.
$\hat\phi(\gamma) = \phi(\gamma)$ for $\gamma \in \dot\Gamma$ (finite-index subgroup)
Let's say that Γ is **strictly* superrigid**.

Not always true:
$G = \mathrm{SL}(2,\mathbb{R})$, $\Gamma \doteq \mathrm{SL}(2,\mathbb{Z})$, $\phi\colon \Gamma \twoheadrightarrow \mathbb{Z} \subseteq G'$.
$\ker\hat\phi \supset \ker\phi$ is infinite, G is simple
$\Rightarrow$ $\hat\phi$ is trivial $\Rightarrow$ $\phi \doteq \hat\phi|_\Gamma$ is trivial. $\rightarrow\!\leftarrow$

Thm. $n > 3 \Rightarrow \mathrm{SL}(n,\mathbb{Z})$ *strictly* superrig in* $\mathrm{SL}(n,\mathbb{R})$.

Want. $\phi\colon \Gamma \to G'$ $\Rightarrow$ ϕ "extends" to $\hat\phi\colon \widetilde{G} \to G'$.

Not always true:
- Γ = lattice in $\mathrm{SL}(4,\mathbb{R})$,
- $\psi\colon \mathrm{SL}(4,\mathbb{R}) \to \mathrm{PSL}(4,\mathbb{R}) = \mathrm{SL}(4,\mathbb{R})/\{\pm I\}$,
- $\psi(\Gamma) \cong \Gamma$ if $I \notin \Gamma$ (torsion-free),
- $\phi\colon \psi(\Gamma) \cong \Gamma$.

$\hat\phi\colon \mathrm{PSL}(4,\mathbb{R}) \to \mathrm{SL}(4,\mathbb{R})$ $\rightarrow\!\leftarrow$

Sometimes need to replace G with a finite cover.

Defn. Γ is **strictly superrigid**: $\phi\colon \Gamma \to G'$
$\Rightarrow \exists\, \hat\phi\colon \widetilde{G} \to G'$, $\hat\phi(\dot\gamma) = \phi(\dot\gamma)$ for $\dot\gamma \in \dot\Gamma$.

Theorem (Margulis Superrigidity Theorem**)**
Γ *is strictly superrigid if* $G \neq \mathrm{SO}(1,n)$, $\mathrm{SU}(1,n)$
and Γ *not cocompact.*

Counterexamples for cocompact lattices:
- $S = \mathrm{diag}(1, 1, -\alpha, -\alpha, -\alpha)$, $\alpha = \sqrt{2}$,
- $G = \mathrm{SO}(S) \cong \mathrm{SO}(2,3)$,
- $\Gamma = G_{\mathbb{Z}[\alpha]}$ is a cocompact lattice in G,
- $\sigma(a + b\alpha) := a - b\alpha$,
- $\sigma\colon \Gamma \to \mathrm{SO}(S^\sigma) \cong \mathrm{SO}(5)$ compact.

$\hat\sigma\colon G \to \mathrm{SO}(5)$ $\rightarrow\!\leftarrow$ $\nexists$ homo $G \to$ compact group.

Margulis Superrigidity Theorem
Γ *is* ***superrigid*** *if* $G \neq \mathrm{SO}(1,n), \mathrm{SU}(1,n)$:
$\phi\colon \Gamma \to G' \Rightarrow \exists\, \hat\phi\colon \widetilde{G} \to G'$, $\hat\phi(\dot\gamma) \in \phi(\dot\gamma)\,C$. ($C$ *cpct*)

Mostow Rigidity Theorem (weak form)
The lattice determines the group:
$\Gamma_1 \cong \Gamma_2 \Rightarrow G_1 \cong G_2$ *if* $Z(G_i) = \{I\}$.

Margulis Arithmeticity Theorem
Every lattice in G *can be obtained by taking* $\mathbb{Z}$*-points*
unless $G \doteq \mathrm{SO}(1,n)$ *or* $\mathrm{SU}(1,n)$.

Warning: Some <u>cocpct</u> latts come from $(G \times \mathrm{cpct})_{\mathbb{Z}}$.

Cor. List of all latts in G (except $\mathrm{SO}(1,n)$, $\mathrm{SU}(1,n)$).

Superrigidity implies arithmeticity

Let Γ be a superrigid lattice in $\mathrm{SL}(n,\mathbb{R})$.
We wish to show $\Gamma \subset \mathrm{SL}(n,\mathbb{Z})$,
i.e., want every matrix entry to be an integer.

First, let us show they are algebraic numbers.
Suppose some $\gamma_{i,j}$ is transcendental.
Then $\exists$ field auto ϕ of $\mathbb{C}$ with $\phi(\gamma_{i,j}) =$???.

Define $\tilde{\phi}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{bmatrix}$.

So $\tilde{\phi}\colon \Gamma \to \mathrm{GL}(n,\mathbb{C})$ is a group homo.
Superrigidity: $\tilde{\phi}$ extends to $\hat{\phi}\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{C})$.
There are uncountably many different ϕ's,
but $\mathrm{SL}(n,\mathbb{R})$ has only finitely many n-dim'l rep'ns.

Γ is a superrigid lattice in $\mathrm{SL}(n,\mathbb{R})$
and every matrix entry is an algebraic number.

Second, show matrix entries are rational.
Recall. Γ is generated by finitely many matrices.
Entries of these matrices generate a field extension of $\mathbb{Q}$ of finite degree. "algebraic number field"
So $\Gamma \subset \mathrm{SL}(n,F)$. For simplicity, assume $\Gamma \subset \mathrm{SL}(n,\mathbb{Q})$

Third, show matrix entries have no denominators.
Actually, show denominators are bounded.
(Then finite-index subgrp has no denoms.)
Since Γ is generated by finitely many matrices,
only finitely many primes appear in denoms.
So suffices to show each prime occurs to bdd power.

Γ is a superrigid lattice in $\mathrm{SL}(n,\mathbb{R})$
and every matrix entry is a rational number.
Show each prime occurs to bdd power in denoms.

This is the conclusion of *p-adic superrigidity*:

Theorem (Margulis)
$\phi\colon \Gamma \to \mathrm{SL}(k,\mathbb{Q}_p) \Rightarrow \phi(\Gamma)$ *has compact closure*
(*unless* $G = \mathrm{SO}(1,n), \mathrm{SU}(1,n)$)
I.e., $\exists \ell$, *no matrix in* $\phi(\Gamma)$ *has* p^ℓ *in denom.*

Summary of proof:

1. $\mathbb{R}$-superrigidity $\Rightarrow$ matrix entries "rational"
2. $\mathbb{Q}_p$-superrigidity $\Rightarrow$ matrix entries $\dot{\in} \mathbb{Z}$

Exercise
Show that if Γ is strictly superrigid in G, then the abelianization of Γ is finite: $\Gamma/[\Gamma,\Gamma]$ is finite.

Exercise
Prove the Mostow Rigidity Theorem under the additional assumption that Γ_1 is strictly superrigid.
Hint: Use a corollary of the Borel Density Theorem to prove that $\hat{\phi}$ is onto.