Lattices in Lie groups 3: Algebraic aspects

- Normal subgrps (center, commutator subgrp, and others)
- Free subgroups and torsion-free subgroups
- Finitely presented and residually finite
- Borel Density Theorem

We ignore finite groups.

Assumps.

- G = simple Lie group $\doteq \mathrm{SL}(n,\mathbb{R})$, $\mathrm{SO}(m,n)$, etc.
- Γ is infinite (i.e., G not compact).
- G is connected (will justify later).

Congruence Subgroups

We prove two basic properties of Γ.

Proposition

*Γ is **residually finite**:*
$\forall \gamma \in \Gamma^{\times}$, $\exists$ homo $\phi\colon \Gamma \to F$ (finite), $\phi(\gamma) \neq e$.

Proof.

$\gamma - \mathrm{I} \neq 0$, so $\exists (\gamma - \mathrm{I})_{ij} \neq 0$. Choose $N \nmid (\gamma - \mathrm{I})_{ij}$.
Ring homo $\mathbb{Z} \to \mathbb{Z}_N$ yields $\mathrm{SL}(n,\mathbb{Z}) \to \mathrm{SL}(n,\mathbb{Z}_N)$.
Let $\rho_N\colon \Gamma \to \mathrm{SL}(n,\mathbb{Z}_N)$ be the restriction to Γ.
Since $\mathbb{Z}_N$ finite, obvious that $\mathrm{SL}(n,\mathbb{Z}_N)$ is finite.
By choice of N, $\rho_N(\gamma)_{ij} \neq \rho_N(\mathrm{I})_{ij}$, so $\rho_N(\gamma) \neq \mathrm{I}$. □

Proposition (Selberg's Lemma**)**

*Γ is **virtually torsion-free**:*
$\exists$ finite-index subgrp $\Gamma' < \Gamma$, Γ' is torsion-free.
(no nontrivial elements of finite order)

Proof.

Define $\rho_3\colon \Gamma \to \mathrm{SL}(n,\mathbb{Z}_3)$ and let $\Gamma' = \ker(\rho_3)$.
It suffices to show Γ' is torsion-free.
Let $\gamma \in \Gamma'$, write $\gamma = \mathrm{I} + 3^k T$, $T \not\equiv 0 \pmod 3$.

$$\begin{aligned}\gamma^m &= (\mathrm{I} + 3^k T)^m \\ &= \mathrm{I} + m(3^k T) + \binom{m}{2} 3^{2k} T^2 + \cdots \\ &\equiv \mathrm{I} + 3^k m T \pmod{3^{k+\ell+1}} \quad \text{if } 3^\ell \mid m \\ &\not\equiv \mathrm{I} \pmod{3^{k+\ell+1}} \quad \text{if } 3^{\ell+1} \nmid m.\end{aligned}$$

□

Finitely generated

Proposition

*Γ is **finitely generated**: $\Gamma = \langle S \rangle$ with S finite.*

Proof for Γ cocompact.

$G = C\Gamma$ with C open, and compact closure.
Let $S = \{\, \gamma \in \Gamma \mid C\gamma \cap C \neq \varnothing \,\}$. ($\Gamma$ discrete, so S finite!)

Let $\gamma \in \Gamma$. Since G is connected, $\exists$ chain
$C, C\gamma_1, C\gamma_2, \ldots, C\gamma_n = C\gamma$, each intersects next.
So $\gamma_1 \in S$, $\gamma_k \gamma_{k-1}^{-1} \in S$, $\gamma\gamma_n^{-1} \in S$.
Hence $\gamma = (\gamma\gamma_n^{-1})(\gamma_n\gamma_{n-1}^{-1}) \cdots (\gamma_2\gamma_1^{-1})\gamma_1 \in \langle S \rangle$. □

In fact, Γ is **finitely presented**:
$\Gamma \cong \langle\, \gamma_1, \gamma_2, \ldots, \gamma_m \mid w_1, w_2, \ldots, w_r \,\rangle$.

Borel Density Theorem

Obs. Classical simple Lie grp $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SO}(m,n)$, ...
is defined by polynomial equations: **Zariski closed.**

Eg. $\mathrm{SL}(n,\mathbb{R}) = \{\, g \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \mid \det(g) = 1 \,\}$
and $\det g$ is a polynomial function of $g_{i,j}$.

$$\det \begin{bmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{1,1} \end{bmatrix} = g_{1,1}\, g_{2,2} - g_{1,2}\, g_{2,1}.$$

Eg. Entry of $g^{\mathsf{T}} \mathrm{I}_{m,n} g$ is sum of $\pm g_{i,j} g_{k,\ell}$. quadratic polynomial
$g^{\mathsf{T}}\, \mathrm{I}_{m,n}\, g = \mathrm{I}_{m,n}$ is system of quadratic equations.

Theorem of real algebraic geometry

Set of real solns of any system of poly eqns (# vars $< \infty$)
has only finitely many connected components.

By ignoring finite group, **assume G is connected**.

Proposition (Borel Density Theorem**)**

Γ is Zariski dense in G:
homo $\rho\colon G \to \mathrm{GL}(n,\mathbb{C})$, $\rho(\Gamma)$ fixes vector v
$\Rightarrow \rho(G)$ fixes v.

Idea of proof for Γ cocompact.

G/Γ compact, $\rho(\Gamma)v = v \Rightarrow \rho(G)v$ bounded.
Let A be any $\mathbb{R}$-split torus of G.
Eigenvalues of elements of A are real.
Since Av bounded, conclude that A fixes v.

Since $\rho(G)$ is simple, it is generated by $\mathbb{R}$-split tori.
So all of $\rho(G)$ fixes v. □

Proposition (Borel Density Theorem)

Homo $\rho\colon G \to \mathrm{GL}(n,\mathbb{C})$

1. $\rho(\Gamma)$ *fixes vector* $v \Rightarrow \rho(G)$ *fixes* v.
2. $\rho(\Gamma)$ *fixes subspace* $W \Rightarrow \rho(G)$ *fixes* W.

Corollary

$Z(\Gamma)$ *is finite.*

Proof.

Define $\rho\colon G \to \mathrm{GL}(\mathrm{Mat}_{k\times k}(\mathbb{R}))$ by $\rho(g)T = gTg^{-1}$.
If $z \in Z(\Gamma)$, then $\rho(\Gamma)z = z$. So $\rho(G)z = z$.
Therefore $z \in Z(G)$. This is a finite group. □

Cor. Γ is not virtually abelian (or solvable).

Normal subgroups

Γ is residually finite.
So ∃ many normal subgroups of finite index.

Theorem (Margulis Normal Subgroups Theorem)

If $\mathrm{rank}_{\mathbb{R}}\, G \geq 2$, *and* N *is a normal subgroup of* Γ, *then either* Γ/N *is finite or* N *is finite.*

Cor. If $\mathrm{rank}_{\mathbb{R}}\, G \geq 2$, then $[\Gamma,\Gamma]$ has finite index in Γ.
I.e., the abelianization of Γ is finite.

Proof.

Suppose not. Then $[\Gamma,\Gamma]$ is finite. So $[\Gamma,\Gamma] \subseteq Z(\Gamma)$.
Therefore, after passing to a finite-index subgroup, we may assume $[\Gamma,\Gamma] = \{\mathrm{I}\}$. So Γ is abelian. →← □

Cor. If $\mathrm{rank}_{\mathbb{R}}\, G \geq 2$, then $[\Gamma,\Gamma]$ has finite index in Γ.

Can replace $\mathrm{rank}_{\mathbb{R}}\, G \geq 2$ with weaker assump that $G \not\approx \mathrm{SO}(1,n), \mathrm{SU}(1,n)$.

Conjecture. For $G = \mathrm{SO}(1,n)$, $[\Gamma',\Gamma']$ has <u>infinite</u> index, for some finite-index subgroup Γ'.

Eg. Torsion-free lattice in $\mathrm{SO}(1,2)$ is free group or surface group.
Abelianization is infinite.

Theorem. *If* $\mathrm{rank}_{\mathbb{R}}\, G = 1$, *then* Γ *has* <u>*many*</u> *normal subgroups of infinite index.*
(*True for every Gromov hyperbolic group.*)

Free subgroups

Proposition

Γ *contains a nonabelian free subgroup.*

Proof is an application of the Ping-Pong Lemma.

Borel Density Theorem

Cor. $\Gamma \not\subseteq$ *connected, proper subgroup* H.

Proof (requires Lie theory).

Let $\mathfrak{h}$ be the Lie algebra of H. This is a subspace of $\mathfrak{g}$.
Since $\Gamma \subseteq H$, we know Γ normalizes H.
So $\mathfrak{h}$ is Γ-invariant for the adjoint representation.
Therefore, $\mathfrak{h}$ must be G-invariant, so $H \triangleleft G$.
Since G is simple, we conclude that $H = G$. □

Cor. Γ is not contained in any Zariski-closed, proper subgroup H of G: Γ is **Zariski dense**.

Proof. H has $< \infty$ components $\Rightarrow H^\circ \dot{\supset} \Gamma$. □

Exercise

Let a be a diagonal matrix in $\mathrm{SL}(n,\mathbb{R})$, and $v \in \mathbb{R}^n$.
If $\{\, a^k v \mid k \in \mathbb{Z} \,\}$ is bounded, show $av = v$.

Exercise

Show that if N is a finite, normal subgroup of Γ, then $N \subseteq Z(G)$.
Hint: Show that the centralizer of N is a finite-index subgrp of Γ, and apply the Borel Density Theorem.