Lecture 2: Geometric aspects

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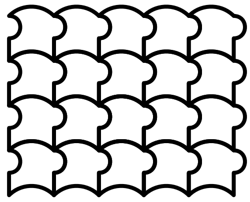
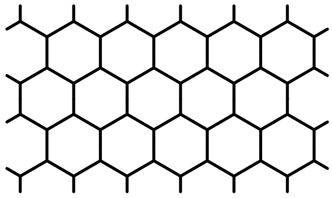
- Symmetric spaces
- classical simple Lie groups
- Real rank and $\mathbb{Q}$-rank

Group theory = the study of symmetry

Example
Symmetries of a **tessellation** (periodic tiling)

symmetry group $\Gamma = \mathbb{Z}^2$ $\qquad \Gamma \doteq \mathbb{Z}^2$

Thm (Bieberbach, 1910). $\forall$ tess of $\mathbb{R}^n$, $\Gamma \doteq \mathbb{Z}^n$.

Other spaces yield groups that are more interesting.

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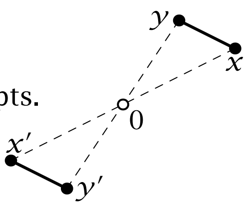
$\mathbb{R}^n$ is a **symmetric space**:
- **homogeneous**: every pt looks like all other pts. $\forall x, y$, $\exists$ isometry $x \mapsto y$.
- reflection through a point ($x' = -x$) is an isometry.

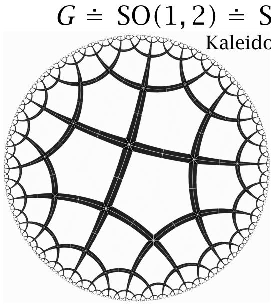
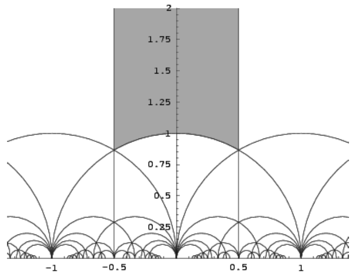


Rem. Let $G = \mathrm{Isom}(X)$. X homog $\Rightarrow$ G transitive $\Rightarrow X = G/K$ with $K = \mathrm{Stab}_G(p)$ cpct.

Symmetry group of tessellation of X is lattice in G if tiles are compact (or finite volume).

Eg. Tess'ns of hyperbolic plane $\mathfrak{H}^2$. (symmetric space)
$G \doteq \mathrm{SO}(1,2) \doteq \mathrm{SL}(2,\mathbb{R})$.

KaleidoTile

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Γ = cocompact lattice $\qquad \Gamma = \mathrm{SL}(2,\mathbb{Z}) \doteq \mathrm{SO}(2,1)_{\mathbb{Z}}$

Tessellations of other symmetric spaces correspond to lattices in other interesting groups.

Other interesting groups G: simple Lie group.
- $\mathrm{SO}(m,n) = \{\, g \in \mathrm{SL}(k,\mathbb{R}) \mid g^{\mathsf{T}}\, \mathtt{I}_{m,n}\, g = \mathtt{I}_{m,n} \,\}$ (lattice: $G_{\mathbb{Z}}$)
- $\mathrm{SU}(m,n)$: change $\mathbb{R}$ to $\mathbb{C}$ and g^{T} to $g^* = \overline{g^{\mathsf{T}}}$ (lattice: $G_{\mathbb{Z}+\mathbb{Z}i}$)
- $\mathrm{Sp}(m,n)$: change $\mathbb{C}$ to $\mathbb{H}$ (lattice: $G_{\mathbb{Z}+\mathbb{Z}i+\mathbb{Z}j+\mathbb{Z}k} = G_{\mathbb{H}_{\mathbb{Z}}}$)
- $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{H})$
- $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{SO}(n,\mathbb{H})$
- finitely many "exceptional grps" (E_6, E_7, E_8, F_4, G_2)

Theorem (Borel and Harish-Chandra**)**
Assume G simple Lie group, defined over $\mathbb{Q}$. Then $G_{\mathbb{Z}}$ is a lattice in G.

Proposition
- *G = simple Lie group* = $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SO}(m,n)$, *etc such that $G^{\mathsf{T}} = G$* (*always true after conjugating*)
- *$K = \{\, k \in G \mid k^{\mathsf{T}} = k^{-1} \,\}$ = max'l compact subgrp,* ($K \doteq \mathrm{SO}(n)$, $\mathrm{SO}(m) \times \mathrm{SO}(n)$, *etc*).

$\Rightarrow$ *G/K is a symmetric space.*

Sketch of proof.
K cpct, so $\exists$ G-inv't Riemannian metric on G/K.
Define $\phi(gK) = (g^{\mathsf{T}})^{-1} K$, so ϕ has order 2.
Average over $\{1, \phi\}$ to make metric ϕ-invariant.
So $\phi \in \mathrm{Isom}(G/K)$.
$\mathtt{I}K$ is an isolated fixed pt of ϕ $\Rightarrow D_{\mathtt{I}K}\phi(v) = -v$
$\Rightarrow \phi$ is the reflection through $\mathtt{I}K$. □

$\mathrm{SL}(2,\mathbb{Z})$ acts on upper half-plane $\mathfrak{h}^2$ by

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * z = \frac{az+b}{cz+d}$.

So $\left\{ \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} * i \right\} = \{ e^{2t} i \} = y$-axis = geodesic.

Therefore, if $gx^t g^{-1}$ is diagonal, and $z = g * i$, then $\{x^t * z\}$ is a geodesic:

Geods in $\mathfrak{h}^2$ ↔ one-param subgrps conj to diag mats.

Similar in G/K:

Prop. Suppose A is an **$\mathbb{R}$-split torus**: connected subgroup of G that is conjugate via $\mathrm{SL}(k,\mathbb{R})$ to a group of diagonal matrices.

Then $\exists p \in G/K$, such that Ap is a **flat**: isometrically embedded copy of $\mathbb{R}^m$ in G/K.

Defn. $\mathrm{rank}_{\mathbb{R}}\, G$ = max dim of **$\mathbb{R}$-split torus.**

Eg. $\mathrm{rank}_{\mathbb{R}}\, \mathrm{SO}(m,n) = \min(m,n) = m$ if $m < n$.

$$x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2 \cong x_1x_{m+1} + x_2x_{m+2} + \cdots + x_m x_{2m} - x_{2m+1}^2 - \cdots - x_{m+n}^2$$

$\mathrm{diag}(t_1, t_2, \ldots, t_m, 1/t_1, 1/t_2, \ldots, 1/t_m, 1, 1, \ldots, 1)$ is an m-dimensional diag'l subgroup of G.

Thm. G/K is Gromov hyperbolic (neg sectional curvature)

$\Leftrightarrow \mathrm{rank}_{\mathbb{R}}\, G = 1$

$\Leftrightarrow G \doteq \mathrm{SO}(1,n), \mathrm{SU}(1,n), \mathrm{Sp}(1,n)$, "$F_4^{-20}$"

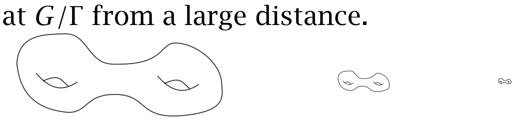
Always: G/K is $\mathrm{CAT}(0)$. (non-positive sectional curvature)

Defn. Let $G = \mathrm{SO}(Q(x))$ and $\Gamma = G_{\mathbb{Z}}$.

- subspace V is **totally isotropic** if $Q(V) = \{0\}$.
  Eg. For $Q_{\mathrm{I}_{3,5}}$: $\{(x,y,z,x,y,z,0,0)\}$.
- $\mathrm{rank}_{\mathbb{Q}}(\Gamma)$ = max dim tot isotrop $\mathbb{Q}$-subspace.

Eg. $\mathrm{rank}_{\mathbb{Q}}(\Gamma) = 0 \Rightarrow Q(x)$ not isotropic over $\mathbb{Q}$ $\Rightarrow G/\Gamma$ compact.

Look at G/Γ from a large distance.



Limit is a point.

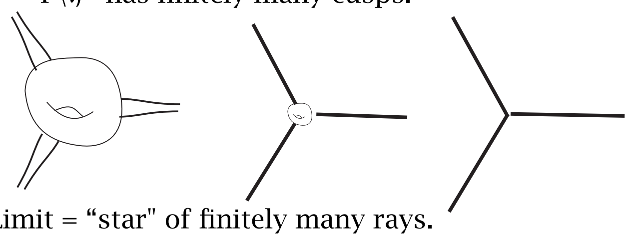
$\therefore$ dimension of limit $= 0 = \mathrm{rank}_{\mathbb{Q}}(\Gamma)$.

This limit is the **asymptotic cone** of G/Γ.

Example

Let $\Gamma = \mathrm{SO}(1,n)_{\mathbb{Z}}$. Then $\Gamma\backslash\mathfrak{h}^n$ not compact.

$\Gamma\backslash\mathfrak{h}^n$ has finitely many cusps.



Limit = "star" of finitely many rays.

$\therefore$ dimension of limit $= 1 = \mathrm{rank}_{\mathbb{Q}}(\Gamma)$.

Theorem (Hattori)

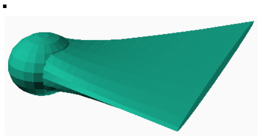
Asymptotic cone of G/Γ (or $\Gamma\backslash G/K$) is a simplicial complex whose dimension is $\mathrm{rank}_{\mathbb{Q}}(\Gamma)$.

Theorem (Hattori)

Dimension of asymptotic cone of G/Γ is $\mathrm{rank}_{\mathbb{Q}}(\Gamma)$.

For general G, $\mathrm{rank}_{\mathbb{Q}}\, G_{\mathbb{Z}}$ = max dim of **$\mathbb{Q}$-split torus**: connected subgroup of G that is conjugate via $\mathrm{SL}(k,\mathbb{Q})$ to a group of diagonal matrices.

Eg. $\mathrm{rank}_{\mathbb{Q}}\, \mathrm{SL}(n,\mathbb{Z}) = n-1$.

$\mathrm{SL}(3,\mathbb{R}) / \mathrm{SL}(3,\mathbb{Z})$: 

The defns of $\mathrm{rank}_{\mathbb{Q}}\, G_{\mathbb{Z}}$ agree when $G = \mathrm{SO}(Q(x))$.

Exer. Assume $G^{\mathsf{T}} = G$ and $K = \{ k \in G \mid k^{\mathsf{T}} = k^{-1} \}$.

Define $\phi(gK) = (g^{\mathsf{T}})^{-1} K$ and let $p = \mathrm{I}K \in G/K$.

Show:

1. ϕ has order 2.
2. p is an isolated fixed point of ϕ:
   If $\phi(g_iK) = g_iK$ for all i, and $g_iK \to p$, then $g_i \in K$ for all large i.
3. $D_p\phi(v) = -v$ for all $v \in T_p(G/K)$.

Assuming that ϕ is an isometry, this implies that ϕ is the reflection through p (see below).

Rem. Let γ be a geodesic through p. Since ϕ is an isometry, we know $\phi \circ \gamma$ is the unique geodesic in the direction $D_p\phi(\gamma'(0)) = -\gamma'(0)$. Therefore $\phi(\gamma(t)) = \gamma(-t)$ for all t (and all γ). This means that ϕ is the reflection through p.