Lattices in Lie groups

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Abstract

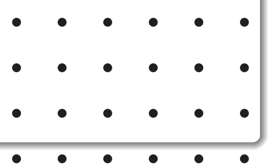
During this mini-course, the students will learn the basic theory of lattices in semisimple Lie groups. Examples will be provided by simple arithmetic constructions. Aspects of the geometric and algebraic structure of lattices will be discussed, and the Superrigidity and Arithmeticity Theorems of Margulis will be described.

Lattices in Lie groups 1: Introduction

- What is a lattice subgroup?
- Arithmetic construction of lattices.
- Compactness criteria.
- Classical simple Lie groups.

A simple example.

- $\mathbb{R}^2$ is a connected Lie group
  - group: $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$
  - Riemannian manifold (metric space, connected)
    distance is right-invariant: $d(x + a, y + a) = d(x, y)$
  - group ops ($x + y$ and $-x$) continuous (differentiable)
- $\mathbb{Z}^2$ is a discrete subgroup (no accumulation points)
- Every point in $\mathbb{R}^2$ is within bdd distance $C = \sqrt{2}$ of $\mathbb{Z}^2$



$\Rightarrow$ $\mathbb{Z}^2$ is a **cocompact lattice** in $\mathbb{R}^2$.

Replace $\mathbb{R}^2$ with interesting group G.

$\mathbb{Z}^2$ is a cocompact lattice in $\mathbb{R}^2$.
Replace $\mathbb{R}^2$ with interesting group G.

Γ is a **cocompact lattice** in G: (discrete subgroup)

- Every pt in G is within bounded distance of Γ.
  $g = c\gamma$ where c is bounded — in cpct set
- $\exists$ compact $C \subseteq G$, $G = C\,\Gamma$.
- G/Γ is cpct. $(g_n\Gamma \to g\Gamma \iff \exists\{\gamma_n\}, g_n\gamma_n \to g)$

Γ is a **lattice** in G:
only require C (or G/Γ) to have finite volume.

Usual way to make a lattice: let $\Gamma = \{\mathbb{Z}\text{-points}\} = G_{\mathbb{Z}}$.

$\mathbb{Z}^2$ is a cocompact lattice in $\mathbb{R}^2$.
Replace $\mathbb{R}^2$ with interesting group G.

Eg. $G = \operatorname{Isom}(\text{hyperbolic } n\text{-space}) \doteq \mathrm{SO}(1, n)$
$\mathfrak{H}^n = \{\, x \in \mathbb{R}^{n+1} \mid x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2 = 1 \,\}^+$

More general: $G = \mathrm{SO}(m, n) = \mathrm{SO}(\mathrm{I}_{m,n})$
where $\mathrm{I}_{m,n} = \operatorname{diag}(1, 1, \ldots, 1, -1, -1, \ldots, -1)$.

Example

$$x^{\mathsf T}\,\mathrm{I}_{1,2}\,x = x^{\mathsf T}\begin{bmatrix}1&&\\&-1&\\&&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = [x_1\ x_2\ x_3]\begin{bmatrix}x_1\\-x_2\\-x_3\end{bmatrix}$$
$$= \begin{bmatrix}x_1^2 - x_2^2 - x_3^2\end{bmatrix} = x_1^2 - x_2^2 - x_3^2.$$

$x^{\mathsf T}\,\mathrm{I}_{1,2}\,x = x_1^2 - x_2^2 - x_3^2$

For $k \times k$ matrix S (invertible, symmetric) and $x \in \mathbb{R}^k$:

- $Q_S(x) := x^{\mathsf T} S\, x$ (quadratic form)
  - $Q_{\mathrm{I}}(x) = x_1^2 + x_2^2 + \cdots x_k^2 = \|x\|^2$
  - $Q_{\mathrm{I}_{m,n}}(x) = x_1^2 + \cdots x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2$
- (special orthogonal group) $\mathrm{SL}(k,\mathbb{R}) = \{\, k \times k, \det = 1 \,\}$
  $\mathrm{SO}(S) := \{\, g \in \mathrm{SL}(k,\mathbb{R}) \mid g^{\mathsf T} S\, g = S \,\}$
  $= \{\, g \in \mathrm{SL}(k,\mathbb{R}) \mid Q_S(gx) = Q_S(x) \,\}$
  $= \mathrm{SO}(Q_S(x))$.

- Q_S is **defined over $\mathbb{Q}$** if coefficients are in $\mathbb{Q}$.
- Q_S is **isotropic over $\mathbb{Q}$** if $\exists v \in (\mathbb{Q}^k)^{\times}$, $Q(v) = 0$.

Exer. $S' = cT^{\mathsf T}ST \Rightarrow \mathrm{SO}(S') = T^{-1}\,\mathrm{SO}(S)\,T \cong \mathrm{SO}(S)$

$\mathbb{Z}^2$ is a cocompact lattice in $\mathbb{R}^2$.
Replace $\mathbb{R}^2$ with interesting group G.
Usual way to make a lattice: let $\Gamma = G_{\mathbb{Z}}$.

Eg. $G = \mathrm{SO}(m,n) = \mathrm{SO}(\mathrm{I}_{m,n}) \Rightarrow G/G_{\mathbb{Z}}$ is <u>not</u> cpct unless $m = 0$ or $n = 0$. (G cpct $\Rightarrow G_{\mathbb{Z}}$ finite $\Rightarrow$ boring)

More general

Assume $G = \mathrm{SO}(Q(x))$ with $Q(x)$ defined over $\mathbb{Q}$.
$G/G_{\mathbb{Z}}$ is cpct $\Leftrightarrow Q(x)$ is <u>not</u> isotropic over $\mathbb{Q}$.

Eg. Here is a cocompact lattice in $\mathrm{SO}(1,2)$:
Let $G = \mathrm{SO}(7x_1^2 - x_2^2 - x_3^2) \cong \mathrm{SO}(1,2)$.
Then $G_{\mathbb{Z}}$ is a cocompact lattice in G. (Since $7 \neq \square + \square$.)

More general

Assume $G = \mathrm{SO}(Q(x))$ with $Q(x)$ defined over $\mathbb{Q}$.
$G/G_{\mathbb{Z}}$ is cpct $\Leftrightarrow Q(x)$ is <u>not</u> isotropic over $\mathbb{Q}$.

Proof (⇒) for $\mathrm{SO}(1,2)$.

$G/G_{\mathbb{Z}}$ compact $\Rightarrow G(\mathbb{Z}^3)^{\times}$ closed $\Rightarrow \vec{0} \notin \overline{G(\mathbb{Z}^3)^{\times}}$.

$\mathrm{SO}(1,2) \cong_{\mathbb{Q}} \mathrm{SO}(Q(x))$ with $Q(x) = x_1x_2 + x_3^2$.

$$\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\mathsf{T}} \mathrm{I}_{1,2} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $a_t = \mathrm{diag}(1/t, t, 1) \Rightarrow Q(a_t x) = Q(x) \Rightarrow a_t \in G$.
- $v := (1,0,0) \in \mathbb{Z}^3$ & $a_t v = (1/t, 0, 0) \to \vec{0}$. □

Fact. $G/G_{\mathbb{Z}}$ always has finite vol: $G_{\mathbb{Z}}$ is a lattice in G.
(if $Q(x)$ is defined over $\mathbb{Q}$)

More general

Assume $G = \mathrm{SO}(Q(x))$ with $Q(x)$ defined over $\mathbb{Q}$.
$G/G_{\mathbb{Z}}$ is cpct $\Leftrightarrow Q(x)$ is <u>not</u> isotropic over $\mathbb{Q}$.

Proof (⇐)

$G/G_{\mathbb{Z}} \hookrightarrow \mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z})$: $gG_{\mathbb{Z}} \mapsto g\,\mathrm{SL}(k,\mathbb{R})$

Proof has 2 parts:

1. $G/G_{\mathbb{Z}}$ is closed in $\mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z})$.
   (because $Q(x)$ is defined over $\mathbb{Q}$)
2. $G/G_{\mathbb{Z}}$ is bounded in $\mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z})$.
   (because $Q(x)$ is not isotropic over $\mathbb{Q}$)

More general

Assume $G = \mathrm{SO}(Q(x))$ with $Q(x)$ defined over $\mathbb{Q}$.
$G/G_{\mathbb{Z}}$ is cpct $\Leftrightarrow Q(x)$ is <u>not</u> isotropic over $\mathbb{Q}$.

Part 1: $G/G_{\mathbb{Z}}$ is closed in $\mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z})$.

Proof.
Suppose $g_n\gamma_n \to g$ with $g_n \in G$ and $\gamma_n \in \mathrm{SL}(k,\mathbb{Z})$.
For $x \in \mathbb{Z}^n$: $Q(\gamma_n x) = Q(g_n\gamma_n x) \to Q(gx)$,
but $Q(\gamma_n x) \in Q(\mathbb{Z}^n) \subset \mathbb{Z}$.
So $Q(\gamma_n x) = Q(gx)$ is eventually constant:
$Q(\gamma_n x) = Q(\gamma_\infty x)$.
Then $Q(gx) = Q(\gamma_n x) = Q(\gamma_\infty x)$.
So $Q(g \cdot \gamma_\infty^{-1} x) = Q(\gamma_\infty \cdot \gamma_\infty^{-1} x) = Q(x)$.
Therefore $g\gamma_\infty^{-1} \in \mathrm{SO}(Q(x)) = G$. □

Part 1: $G/G_{\mathbb{Z}}$ is closed in $\mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z})$.

Part 2: $G/G_{\mathbb{Z}}$ is bounded in $\mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z})$.

Lemma (Mahler Compactness Criterion**)**
Let $C \subset \mathrm{SL}(k,\mathbb{R})$.
The image of C *in* $\mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z})$ *is bounded*
$\iff$ $\vec{0}$ *is not an accumulation point of* $C\,\mathbb{Z}^n$.

Proof (⇒).
Spse $c_n z_n \to \vec{0}$ and $c_n\gamma_n \to h$. Since $h\mathbb{Z}^n$ is discrete, and $h(\gamma_n^{-1} z_n) \approx c_n z_n \approx \vec{0}$, we have $z_n = \vec{0}$. □

Converse is an exercise (for $k = 2$*).*

Part 2 of the proof.
$g_n z_n \in G(\mathbb{Z}^n)^{\times} \Rightarrow Q(g_n z_n) = Q(z_n) \in Q((\mathbb{Z}^n)^{\times}) \subseteq \mathbb{Z}^{\times}$
$\Rightarrow Q(g_n z_n) \not\to 0 \quad \Rightarrow g_n z_n \not\to 0$. □

More general

Assume $G = \mathrm{SO}(Q(x))$ with $Q(x)$ defined over $\mathbb{Q}$.
$G/G_{\mathbb{Z}}$ is cpct $\Leftrightarrow Q(x)$ is <u>not</u> isotropic over $\mathbb{Q}$.

Eg. $7x_1^2 - x_2^2 - x_3^2$ is not isotropic over $\mathbb{Q}$.

Theorem of Number Theory
$Q(x)$ isotropic over $\mathbb{R}$ with at least 5 variables
$\Rightarrow Q(x)$ is isotropic over $\mathbb{Q}$.

Generalization of fact that every positive integer is a sum of 4 squares.

So *More general* never provides a cocompact lattice in $\mathrm{SO}(m,n)$ with $m + n \geq 5$ (unless $m = 0$ or $n = 0$).

Cocompact lattices in SO(m, n)

- $Q(x) = x_1^2 + x_2^2 - \alpha x_3^2 - \alpha x_4^2 - \alpha x_5^2, \quad \alpha = \sqrt{2},$
- $G = \mathrm{SO}(Q(x)) \cong \mathrm{SO}(2,3),$
- $\Gamma = G_{\mathbb{Z}[\alpha]} = (\mathbb{Z} + \mathbb{Z}\alpha)$*-points.*

Then Γ *is a cocompact lattice in* G*.*

Idea of proof. $\sigma(a + b\alpha) := a - b\alpha$. (Galois aut)
$G^\sigma := \mathrm{SO}(Q^\sigma) = \mathrm{SO}(x_1^2 + x_2^2 + \alpha x_3^2 + \alpha x_4^2 + \alpha x_5^2) \cong \mathrm{SO}(5)$.
Map $\omega \mapsto (\omega, \omega^\sigma)$ embeds $\mathbb{Z}[\alpha] \hookrightarrow \mathbb{R} \oplus \mathbb{R}$.
$(1, 1^\sigma)$, (α, α^σ) are lin indep so image discrete.
So image of Γ in $G \times G^\sigma$ is discrete. Cocpct lattice!
$Q(x)$ not isotropic over $\mathbb{Q}[\alpha]$:
$Q(v) = 0 \Rightarrow Q^\sigma(v^\sigma) = 0 \Rightarrow v^\sigma = 0 \Rightarrow v = 0$.
Can mod out compact group G^σ. □

Replace $\mathbb{R}^2$ *with interesting* G*:* simple Lie group.

- $\mathrm{SO}(m,n) = \{\, g \in \mathrm{SL}(k,\mathbb{R}) \mid g^{\mathsf{T}}\, \mathrm{I}_{m,n}\, g = \mathrm{I}_{m,n} \,\}$ (lattice: $G_{\mathbb{Z}}$)
- $\mathrm{SU}(m,n)$: change $\mathbb{R}$ to $\mathbb{C}$ and g^{T} to $g^* = \overline{g^{\mathsf{T}}}$ (lattice: $G_{\mathbb{Z}+\mathbb{Z}i}$)
- $\mathrm{Sp}(m,n)$: change $\mathbb{C}$ to $\mathbb{H}$ (lattice: $G_{\mathbb{Z}+\mathbb{Z}i+\mathbb{Z}j+\mathbb{Z}k} = G_{\mathbb{H}_{\mathbb{Z}}}$)
- $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{H})$
- $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{SO}(n,\mathbb{H})$
- finitely many "exceptional grps" $(E_6, E_7, E_8, F_4, G_2)$

Theorem (Borel and Harish-Chandra**)**
Assume G *simple Lie group, defined over* $\mathbb{Q}$*.*
Then $G_{\mathbb{Z}}$ *is a lattice in* G*.*

Exercise
Assume Γ is a lattice in G.

1. Show that every finite-index subgroup of Γ is a lattice in G.
2. Show that if Γ is a cocompact lattice in G, then every finite-index subgroup of Γ is a cocompact lattice in G.
3. Show that the following are equivalent:
   1. G is compact.
   2. Γ is finite.
   3. $\mathrm{vol}(G)$ is finite.

Hint: You may assume the following basic facts about volume (for subsets of G):

- $\mathrm{vol}(Eg) = \mathrm{vol}(E)$ for all $g \in G$.
- If E is compact, then $\mathrm{vol}(E) < \infty$.
- If U is open and nonempty, then $\mathrm{vol}(U) > 0$.
- If $E \subseteq F$, then $\mathrm{vol}(E) \le \mathrm{vol}(F)$.
- $\mathrm{vol}(E \cup F) \le \mathrm{vol}(E) + \mathrm{vol}(F)$.
- If E and F are disjoint, and are either open or closed, then $\mathrm{vol}(E \cup F) = \mathrm{vol}(E) + \mathrm{vol}(F)$.

Furthermore, G is **locally compact**. This means that every closed ball $B_r(g)$ is compact.

Rem. Since G has a lattice, we also have $\mathrm{vol}(gE) = \mathrm{vol}(E)$ for all $g \in G$. However, for some Lie groups, it is only true that $\mathrm{vol}(E) = \mathrm{vol}(Eg)$ (or only that $\mathrm{vol}(E) = \mathrm{vol}(gE)$), not both.

Exercise
Prove $(\Leftarrow)$ of Mahler Compactness Criterion for $k{=}2$.

Hint: Let $\{c_n\}$ be a sequence of points in C.
Since $C \subseteq \mathrm{SL}(2,\mathbb{R})$, there is a sequence $\{\gamma_n\}$ in $\mathrm{SL}(2,\mathbb{Z})$, such that $c_n \gamma_n \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ is bounded. (Why?)
By passing to a subsequence, we may assume $c_n \gamma_n \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ converges to some $v_1 \in \mathbb{R}^2$.
Note that $v_1 \neq 0$. (Why?)
Now show there is a sequence $\{\gamma_n'\}$ in $\left[\begin{smallmatrix}1 & *\\0 & 1\end{smallmatrix}\right]$, such that $c_n \gamma_n \gamma_n' \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ converges to some $v_2 \in \mathbb{R}^2$.
This implies $c_n \gamma_n \gamma_n' \to [v_1\ v_2]$.
So $\{c_n\, \mathrm{SL}(2,\mathbb{Z})\}$ has a convergent subsequence.

Definition
An element u of $\mathrm{SL}(k,\mathbb{Q})$ is **unipotent** if the following equivalent conditions are true:

1. 1 is the only eigenvalue of u (in $\mathbb{C}$).
2. $(u - \mathrm{I})^k = 0$.
3. u is conjugate in $\mathrm{SL}(k,\mathbb{Q})$ to matrix that is upper-triangular with only 1s on the diagonal.

Exercise
Assume $G = \mathrm{SO}(Q_S(x))$, with Q_S defined over $\mathbb{Q}$. Show that if $G_{\mathbb{Z}}$ has a unipotent element other than I, then $G/G_{\mathbb{Z}}$ is not compact.

Hint: Find $u \in G$ and $v_1, v_2 \in (\mathbb{Q}^k)^\times$, such that $uv_1 = v_1$ and $uv_2 = v_1 + v_2$. Also note that $(uv)^{\mathsf{T}} S\,(uw) = v^{\mathsf{T}} S\, w$.