Surfaces containing two circles through each point

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joint work with

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Perspectives in real geometry

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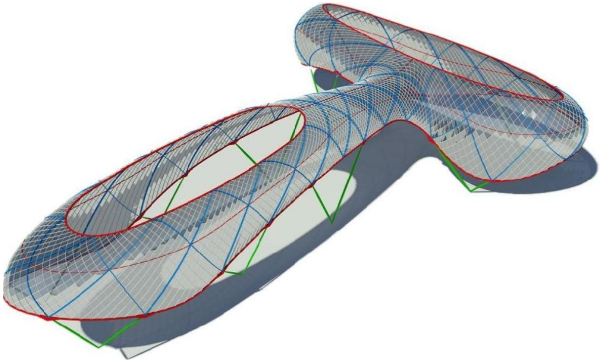
Overview

1. A trailer: an elementary motivation
2. A theorem: examples and the statement
3. The proof: a general plan

1
A trailer:
an elementary motivation

Motivation

Rationalization is approximation of a design by a form suitable for actual fabrication



Rationalization is approximation of a design by a form suitable for actual fabrication

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Rationalization is approximation of a design by a form suitable for actual fabrication

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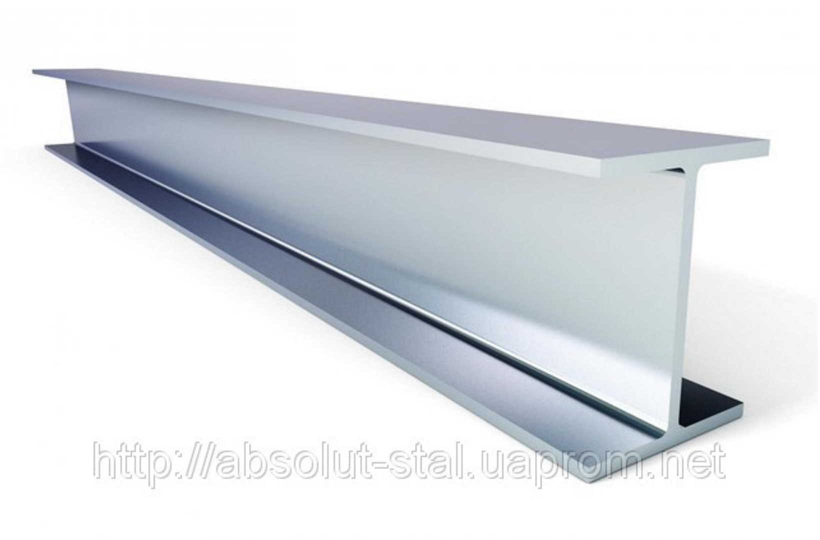
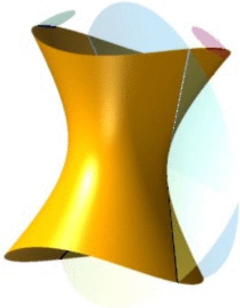
The simplest building block



Vladimir Grigoryevich Shukhov (1896)

One-sheeted hyperboloid of revolution is the result of revolution of a line about an axis, not in one plane with the line

One-sheeted hyperboloid is the result of its dilatation in one direction



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Let 2 points move uniformly along 2 lines, not in one plane. Then the line through the points draws a *hyperbolic paraboloid*.

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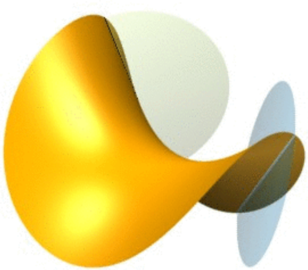
A hyperbolic paraboloid contains 2 lines through each point



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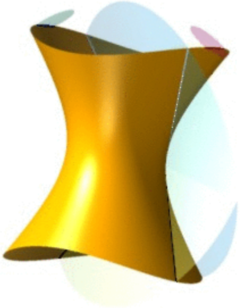
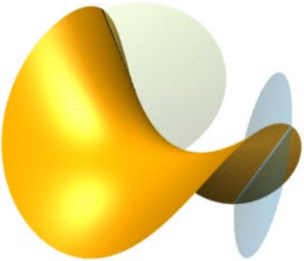
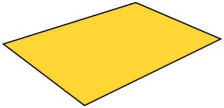
All surfaces containing two lines through each point

All surfaces containing 2 lines through each point:







What if beams have form of circular arcs?

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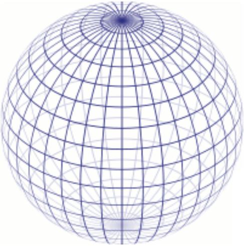
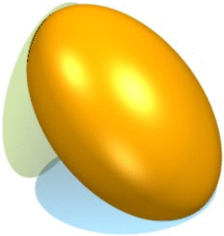
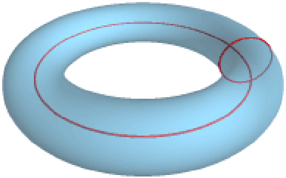
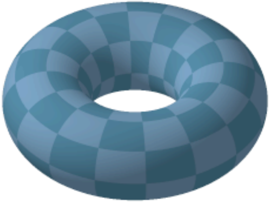
Folklore examples (Hilbert–Cohn-Vossen, 1932)







Villarceau circles — section of a torus by a plane touching the torus at 2 points



Villarceau circles (XIX c.)
in Strasburg Cathedral (XII-XV c.):

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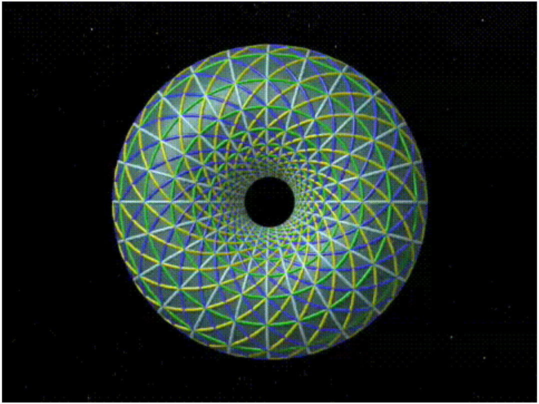
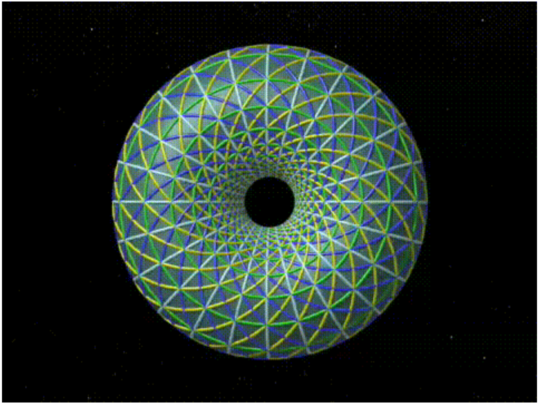
A torus contains 4 circles through each point



The image of a torus under an inversion contains 4 circles through each point



2
A theorem: examples and the statement

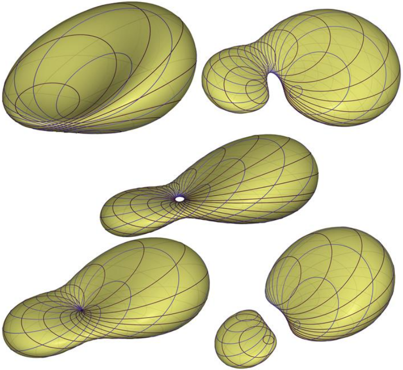
A *Darboux cyclide* is given by the equation

$$Q(x, y, z, x^2 + y^2 + z^2) = 0,$$

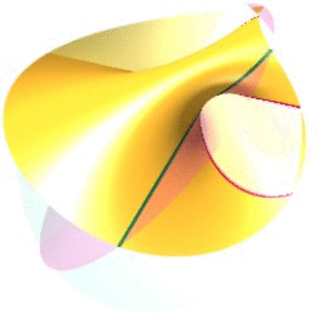
where

$Q \in \mathbb{R}[x, y, z, t]$,

$\deg Q = 2$ or 1.



Almost each Darboux cyclide contains ≥ 2 circles through each point



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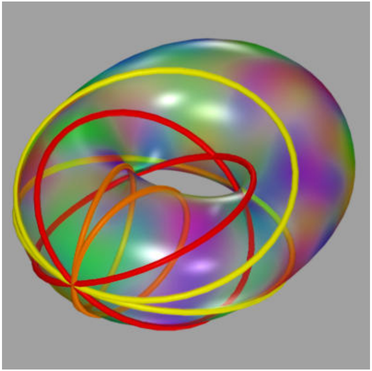
Some Darboux cyclides contain 6 circles through each point (R. Blum, 1980)



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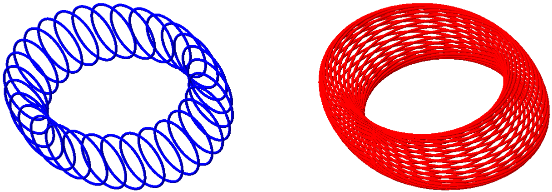
Theorem. A smooth surface containing

- 7 transversal circlular arcs through each point is a *sphere* **(N.Takeuchi,1995);**
- 3 or 2 *cospheric* or 2 *orthogonal* transversal circlular arcs through each point is a *Darboux cyclide* **(N.Lubbes,2014, J.Coolidge,1906, T.Ivey,1995);**

Example (H. Pottmann, 2010).

Translation of a circle along another circle:

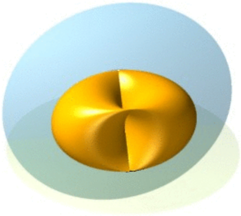


$$\{\, p+q : p \in A, q \in B \,\},$$

where $A, B \subset \mathbb{R}^3$ are circles. *Not a cyclide!*

Example (H. Pottmann, 2010).

Translation of a circle along another circle:



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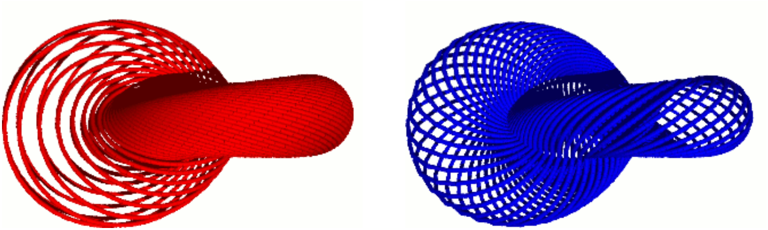
$$\{\, p + q : p \in A, q \in B \,\}$$

where $A, B \subset \mathbb{R}^3$ are circles.

Example (S. Żube, 2011). The surface



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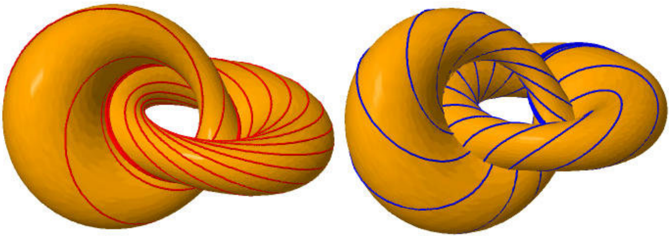
$$\left\{ 2\frac{p \times q}{|p+q|^2} : p \in A, q \in B \right\},$$

where $A, B \subset S^2$ are circles.

Example (S. Żube, 2011).

= the stereographic projection of the surface



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$\{ p \cdot q : p \in A, q \in B \}$ (*quaternion product*!),

where $A, B \subset S^2$ are circles.

Theorem (N.Lubbes,2014). An algebraic surface in S^3 containing a *great* circle and another circle through each point is *Clifford translational* or the inverse stereographic projection of *Darboux cyclide*.

Theorem (J.Kollár,2016). An algebraic surface in S^n containing *infinitely* many transversal circles through each point is a *sphere* or a *Veronese surface*. All Veronese surfaces in S^n are Möbius equivalent.

By an *analytic surface* in $\mathbb{R}^n$ we mean the image of an injective real analytic map of a planar domain into $\mathbb{R}^n$ with everywhere nondegenerate differential.

A circular arc *analytically depending* on a point is a real analytic map of a planar domain into the variety of all circular arcs in $\mathbb{R}^n$.

Theorem (S.'15). If through each point of an analytic surface in $\mathbb{R}^3$ one can draw two transversal circular arcs fully contained in the surface (and analytically depending on the point) then some composition of inversions takes the surface to a subset of one of the following sets:

- a Darboux cyclide, or
- a Euclidean translational surface, or
- a Clifford translational surface.

3
The proof: a general plan

Step 1: reduction of finding surfaces in S^n to parametrization of Pythagorean $(n+2)$-tuples;

Step 2: parametrization of Pythagorean 6-tuples of small degree; this gives surfaces in S^4;

Step 3: extraction of surfaces in $\mathbb{R}^3$ from the obtained set of surfaces in S^4.

Remark (J. Schicho, 2000). A surface in $\mathbb{C}P^n$ containing *2 conic sections* through almost each point has a parametrization

$$\Phi(u, v) = X_1(u, v) : \cdots : X_{n+1}(u, v),$$

where $X_1, \ldots, X_{n+1}$ have degree at most 2 in each variable u and v.

Theorem (Krasauskas–S., 2015). Assume that through each point of an analytic surface in S^{n-2} one can draw two noncospheric circular arcs fully contained in the surface (and analytically depending on the point). Assume that through each point in some dense subset of the surface one can draw only finitely many circular arcs fully contained in the surface. Then the surface (possibly besides a one-dimensional subset) has a parametrization

$$\Phi(u,v) = X_1(u,v) : \dots : X_n(u,v),$$

where $X_1, \dots, X_n \in \mathbb{R}[u,v]$ have degree at most 2 in each variable u and v and satisfy the equation

$$X_1^2 + \dots + X_{n-1}^2 = X_n^2 \qquad (1)$$

Problem on Pythagorean n-tuples.
Solve

$$X_1^2 + \cdots + X_{n-1}^2 = X_n^2$$

in polynomials of degree at most 2 in each variable u and v.

- $n = 3$: Complete parametrization
  $X_1 = 2ABD$, $X_2 = (A^2 - B^2)D$, $X_3 = (A^2 + B^2)D$
- $n = 4$: Complete parametrization (Dietz et al., 1993)
- $n = 6$: Partial -//- (Kocik, 2007)
- $n = 6$ and 1 variable: still accessible
- $n = 6$, 2 variables, $\deg 2$: (Kollar,2016)
- $n = 6$, 2 variables, $\deg 4$: 1st hard case
- $n = 5$: even harder.

Some notation

A *Möbius transformation* is a linear transformation $\mathbb{R}^6 \to \mathbb{R}^6$ (not depending on the variables u, v) which preserves (1).

$\mathbb{H}_{mn} \subset \mathbb{H}[u, v]$ is the set of polynomials with quaternionic coefficients of degree at most m in u and at most n in v

(the *variables commute* with everything)

$\mathbb{R}_{mn} \subset \mathbb{R}[u, v]$ is defined analogously

Theorem (S., 2015). Polynomials $X_1, \dots, X_6 \in \mathbb{R}_{22}$ satisfy $X_1^2 + \dots + X_5^2 = X_6^2$ if and only if up to Möbius transformation we have

$$X_1 + iX_2 + jX_3 + kX_4 = 2ABCD,$$
$$X_5 = (|B|^2 - |AC|^2)D,$$
$$X_6 = (|B|^2 + |AC|^2)D$$

for some $A, B, C \in \mathbb{H}_{11}$, $D \in \mathbb{R}_{22}$ such that $|B|^2D, |AC|^2D \in \mathbb{R}_{22}$.

Remark.

Stereographic projection $S^4 \to \mathbb{R}^4 = \mathbb{H}$,
$X_1 : \dots : X_6 \mapsto (X_1, \dots, X_4)/(X_6 - X_5)$,
gives

$$\Phi(u,v) = \bar{A}(u,v)^{-1} B(u,v) \bar{C}(u,v)^{-1},$$

where $A, B, C \in \mathbb{H}_{11}$ and $AC \in \mathbb{H}_{11}$ — *quaternionic fraction-linear expression* in both u and v.

Theorem (Krasauskas–S., 2015)

If the surface

$$\Phi(u,v) = A(u,v)^{-1}B(u,v)C(u,v)^{-1},$$

where $A, B, C \in \mathbb{H}_{11}$ and $AC \in \mathbb{H}_{11}$, is contained in $\mathbb{R}^3$ (respectively, in S^3) then it is a subset of either Euclidean (respectively, Clifford) translational surface or a Darboux cyclide (respectively, an intersection of S^3 with another 3-dimensional quadric).

Example. Let $X, Y, Z \in \mathbb{R}[u, v]$.

$X^2 + Y^2 = Z^2 \implies$

$(X + iY)(X - iY) = Z^2 \overset{\text{unique factorization}}{\implies}$

$X + iY = C^2 D, Z = |C|^2 D$

for some $C \in \mathbb{C}[u, v]$, $D \in \mathbb{R}[u, v] \implies$

$$X = (A^2 - B^2)D,$$
$$Y = 2ABD,$$
$$Z = (A^2 + B^2)D,$$

where $A = \mathrm{Re}\,C$, $B = \mathrm{Im}\,C$.

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Denote:

$$Q := X_1 + iX_2 + jX_3 + kX_4,$$
$$P := X_6 - X_5,$$
$$R := X_6 + X_5.$$

Then:

- $X_1^2 + \cdots + X_5^2 = X_6^2 \Leftrightarrow \bar{Q}Q = PR$;
- the required parametrization is $(P, Q, R) = (2|AC|^2D, 2ABCD, 2|B|^2D)$.

Denote:

$$\begin{aligned} Q &:= X_1 + iX_2 + jX_3 + kX_4, \\ P &:= X_6 - X_5, \\ R &:= X_6 + X_5. \end{aligned}$$

Then:

- $X_1^2 + \cdots + X_5^2 = X_6^2 \Leftrightarrow \bar{Q}Q = PR$;
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Remark. In the *unique factorization domain* $\mathbb{C}[u, v]$ all solutions of the system

$$Q\bar{Q} = PR, \bar{P} = P, \bar{R} = R$$

are parametrized by

$$(P, Q, R) = (A\bar{A}D, ABD, B\bar{B}D), \quad \bar{D} = D.$$

Remark. $\mathbb{H}[u]$ is a *unique factorization domain* in a sense **(Ore, 1933)**.

Example (Beauregard, 1993).

$$Q_B := u^2v^2 - 1 + (u^2 - v^2)i + 2uvj$$

is *irreducible* in $\mathbb{H}[u, v]$ but

$$|Q_B|^2 = \overbrace{(u^2 - \sqrt{2}u + 1)(v^2 - \sqrt{2}v + 1)}^{P_B} \times \times \underbrace{(u^2 + \sqrt{2}u + 1)(v^2 + \sqrt{2}v + 1)}_{R_B}.$$

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Main idea: parametrization up to a *"Möbius transformation"*

$$(R, Q, P) \mapsto (R, Q - TR, P - T\bar{Q} - Q\bar{T} + TR\bar{T}),$$

where $T \in \mathbb{H}$ (preserves the Eq. $\bar{Q}Q = PR$)

Example. We have

- $R_B = |B|^2$;
- $Q_B = ABC - T|B|^2$;
- $P_B = |AC|^2 - ABC\bar{T} - T\bar{C}\,\bar{B}\,\bar{A} + T|B|^2\bar{T}$,

where

- $A = (1-j)(u + \frac{-i-j}{\sqrt{2}})$,
- $B = (v + \frac{1+k}{\sqrt{2}})(u + \frac{1+i}{\sqrt{2}})$,
- $C = v + \frac{-j-k}{\sqrt{2}}$,
- $T = j$.

Splitting Lemma (Krasauskas–S.'15).
If $|Q(u,v)|^2 = P(v)R(u)$ for some $Q \in \mathbb{H}_{11}$, $P \in \mathbb{R}_{02}$, $R \in \mathbb{R}_{20}$ then either $Q(u,v) = A(u)B(v)$ or $Q(u,v) = B(v)A(u)$ for some $A \in \mathbb{H}_{10}$, $B \in \mathbb{H}_{01}$.

Proof. Assume that $\deg P = \deg R = 2$; otherwise Q does not depend on one of the variables and there is nothing to prove. Expand
$Q(u,v) =: Q_0(u) + vQ_1(u) =: Q_{00} + Q_{10}u + Q_{01}v + Q_{11}uv$.
We have $Q_{11} \neq 0$. Take $q \in \mathbb{H}$ such that $Q_0(u) + qQ_1(u)$ is a constant and denote the constant by p; that is, set
$q := -Q_{10}Q_{11}^{-1}$ and $p := Q_0 + qQ_1 = Q_{00} - Q_{10}Q_{11}^{-1}Q_{01}$.
Consider the polynomial $|Q|^2(u,q)$ obtained by substitution of the quaternion q into the *real* polynomial $|Q|^2(u,v)$. On one hand, $|Q|^2(u,q) = P(q)R(u)$ is divisible by $R(u)$ of degree 2.
On the other hand,
$|Q|^2(u,q) = q(qQ_1 + Q_0)\bar{Q}_1 + (qQ_1 + Q_0)\bar{Q}_0 = qp\bar{Q}_1 + p\bar{Q}_0$
has degree ≤ 1. Thus $|Q|^2(u,q) = qp\bar{Q}_1 + p\bar{Q}_0 = 0$ identically.
Now for $p \neq 0$ we get $Q_0 = -Q_1\bar{p}\,\bar{q}\,\bar{p}^{-1}$, hence
$Q = Q_1(u)(v - \bar{p}\,\bar{q}\,\bar{p}^{-1})$ as required. For $p = 0$ we get
$Q_0 = -qQ_1$, hence $Q = Q_0 + vQ_1 = (v-q)Q_1(u)$ as required.

An ***n*** $\times$ ***n*** *grid* is two collections of $n+1$ disjoint arcs intersecting at $(n+1)^2$ points.

Problem (curved chess-board). Is each 8×8 grid of circular arcs contained in a surface with 2 circles through each point?

Problem (constant radii or angle).
Let α, r, and R be fixed.
Find all surfaces in $\mathbb{R}^3$ containing 2 circles *of radii r and R* or *intersecting at angle α* through each point.

References

M. Skopenkov, R. Krasauskas, *Surfaces containing two circles through each point*, submitted, under revision **http://arxiv.org/abs/1503.06481**

THANKS!