A local Harnack problem (after J.-J. Risler)

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Contents

Harnack and Thom-Smith inequalities

The key fact of the topology of real algebraic curves

Theorem (Harnack, 1876)

For any $d \geq 1$,

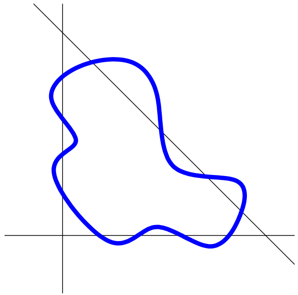
(1) *the number n of connected components of the real point set $\mathbb{R}C$ of a smooth real algebraic curve $C \subset \mathbb{P}^2$ of degree d does not exceed*

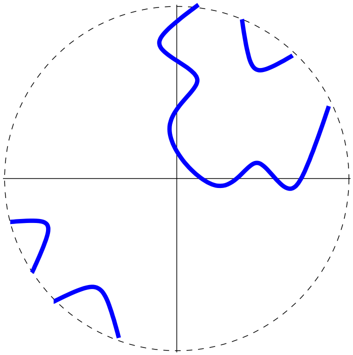
$$\frac{(d-1)(d-2)}{2} + 1 \,;$$

(2) *for each $1 \leq n \leq \frac{(d-1)(d-2)}{2} + 1$ there exists a smooth real curve C of degree d with n connected components of $\mathbb{R}C$.*

Remark: The general Klein's upper bound $\quad n \leq g + 1$.

Example: The Harnack M-curve





and $\frac{d(d-3)}{2}$ extra ovals

The Thom-Smith inequality

Theorem (Smith)

Let $c : X \to X$ be a continuous involution, and $H_*(X, \mathbb{Z}/2)$ is finite. Then

$$b_*(\mathrm{Fix}(c), \mathbb{Z}/2) \leq b_*(X, \mathbb{Z}/2) \ .$$

Corollary (Thom)

Any real (affine or projective) algebraic variety V satisfies

$$b_*(\mathbb{R}V, \mathbb{Z}/2) \leq b_*(V, \mathbb{Z}/2) \ .$$

Note that $b_*(C, \mathbb{Z}/2) = 2g + 2$ for a smooth projective curve of genus g.

Definition

M-object means equality in the Smith-Thom relation.

Sharpness of the global Thom-Smith inequality

Theorem (Itenberg-Viro)

For any $n \geq 2$ and $d \geq 1$ there exists a smooth real M-hypersurface of degree d in $\mathbb{P}^n$.

Remark: B. Bertrand extended this to hypersurfaces and complete intersections in certain toric varieties and also exhibited examples of non-sharpness.

Theorem (Kharlamov-Risler-Shustin)

For any lattice triangle $\Delta \subset \mathbb{R}^2$ there exists a real polynomial $F \in \mathbb{R}[x, y]$ with Newton triangle Δ that defines an M-curve $C \subset \mathrm{Tor}(\Delta)$ which regularly intersects the toric divisors in only real points.

The local Harnack problem

Let $(C, z) \subset \mathbb{C}^2$ be a real isolated curve singularity, $B(C, z) \subset \mathbb{C}^2$ its Milnor ball. Denote by μ the Milnor number, by r the number of branches of (C, z), and by $r_{\mathbb{R}}$ the number of real branches.
A **smoothing** of (C, z) is a real analytic deformation $C_t \subset B(C, z)$, $0 \le t < \varepsilon$, of $C = C_0$ such that all curves C_t, $t \neq 0$, are real, smooth and intersect $\partial B(C, z)$ transversally.
Each set $\mathbb{R}C_t$, $t \neq 0$, contains $r_{\mathbb{R}}$ segments and and some number v of circles (ovals), which does not depend on $t \neq 0$.

Problem: *(V. I. Arnold, 1976)*
Determine the maximal possible number of ovals for smoothings of a given real plane curve singularity.

The first answer:
J.-J. Risler. Une analogue local du théorème de Harnack. *Invent. Math.* **89** (1987), 119–137.

Invent. math. 89, 119–137 (1987)

Inventiones mathematicae

Un analogue local du théorème de Harnack

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§ 0. Introduction

Le «Théorème de Harnack» dit que toute courbe algébrique réelle lisse de degré d dans $\mathbb{R}\mathbb{P}^2$ possède au plus $(d-1)(d-2)/2+1$ composantes connexes, et que l'on peut pour tout degré $d \geqq 0$ construire une courbe de degré d ayant ce nombre maximal de composantes.

Let but de cet article est d'étudier l'analogue local de ce théorème sous la forme suivante: quels sont les types topologiques des déformations locales d'un germe (analytique) de courbe plane donné?

A'Campo ([A]) a étudié un problème similaire: il a montré en particulier qu'un germe de courbe plane réelle peut toujours être déformé de façon à acquérir le nombre maximum possible de points doubles réels (à savoir $1/2(\mu + r - 1)$; cf. § 1 pour les notations).

On montre ici qu'un germe en 0 de courbe plane irréductible réelle peut toujours être déformé en une courbe lisse ayant $\mu/2+1$ composantes connexes au voisinage de 0 (théorème 6.1); dans le cas réductible, le résultat analogue n'est montré que lorsque les germes des composantes irréductibles ont des tangentes distinctes (théorème 8.2).

Les § 1 à 5 ont un caractère introductif.

Cet article doit beaucoup à R. Benedetti, B. Chevallier et A. Marin; je les en remercie.

§ 1. Rappels sur les germes analytiques

Soit $\mathbb{R}\{X, Y\}$ l'anneau des germes à l'origine de $\mathbb{R}^2$ des fonctions analytiques de deux variables réelles.

Nous ferons systématiquement au cours de cet article un abus de language consistant à identifier un germe de courbe dans $\mathbb{R}^2$ et un représentant bien choisi de ce germe.

Par exemple si $f \in \mathbb{R}\{X, Y\}$ est le germe d'une fonction analytique, nous

Upper bounds

Pick some $0 < t < \varepsilon$ and introduce the spaces $M = C_t$ (Milnor fiber), $\widehat{M}$ obtained from M by attaching a disc to each hole corresponding to a non-real branch of (C, z), and $\widetilde{M}$ obtained from M by attaching a disc to each hole of M. The Thom-Smith inequality applied to these surfaces respectively yields

(1) $v \leq \frac{1}{2}(\mu - r_{\mathbb{R}} + 1)$

(2) **local Harnack bound**

$$v \leq \frac{1}{2}(\mu - r + 1) \text{ if } r_{\mathbb{R}} > 0, \quad v \leq \frac{1}{2}(\mu - r + 3) \text{ if } r_{\mathbb{R}} = 0$$

(3) **refined local Harnack bound**

$$v \leq \frac{1}{2}(\mu - r + 3) - l\ ,$$

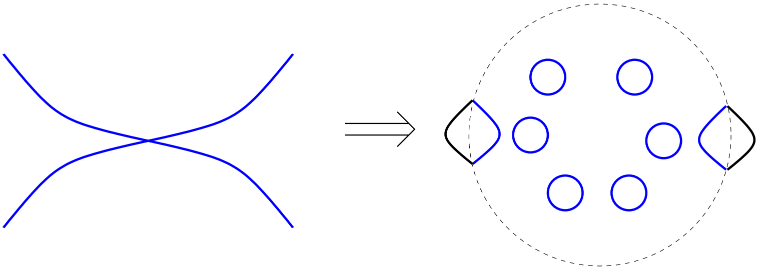
where l is the number of circles obtained from the segments of $\mathbb{R}C_t$ by joining with an arc each pair of endpoints corresponding to the same real branch of C_t

Definition

***M*-smoothing** - equality in the local Harnack bound

Weak *M*-smoothing - equality in the refined local Harnack bound

Example: Weak M-smoothing of the two-cuspidal singularity $y^4 - x^6 = 0$



$\mu = 15,\ r = 2$

$l = 2$

$v = \frac{\mu - r + 3}{2} - l = 6$

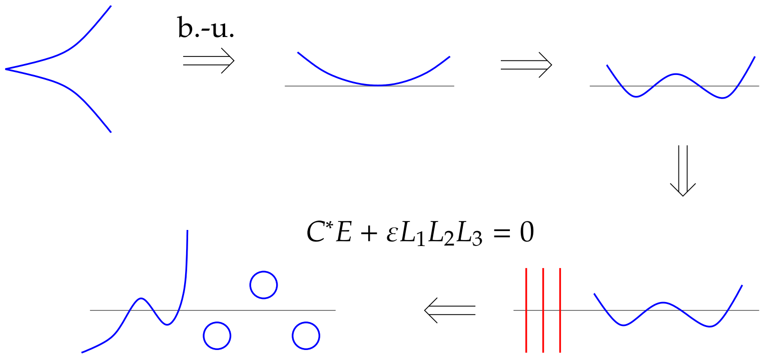
Theorem (Risler, 1987)

Let (C, z) *be irreducible, i.e.,* $r = r_{\mathbb{R}} = 1$*. Then it admits an* M*-smoothing, i.e., a smoothing with* $v = \frac{1}{2}\mu$ *ovals.*

Sketch of the proof. Inductive blow-up construction based on the following

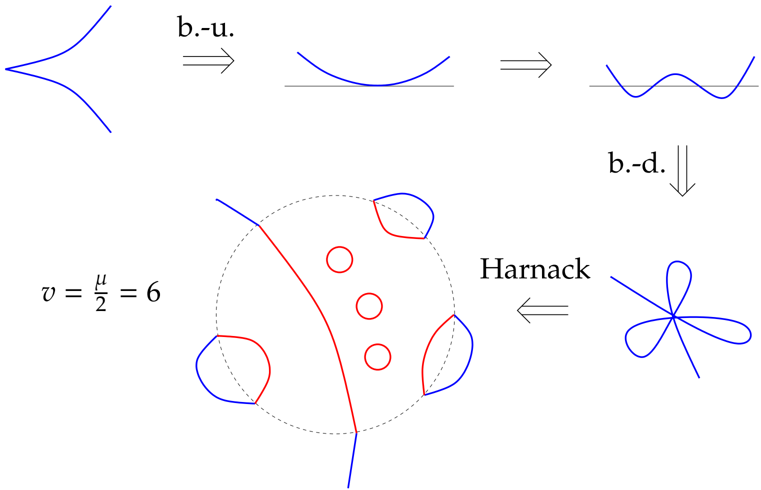
Claim: *Let* L_1, L_2 *be two distinct real lines through* z*. Then there exists an* M*-smoothing of* (C, z) *containing a segment regularly intersecting* L_1 *in* $(C \cdot L_1)$ *points and* L_2 *in* $(C \cdot L_2)$ *points.*

We illustrate the construction in the example of singularity $y^4 - x^5 = 0$ with $\frac{1}{2}\mu = 6$



$$v = 3 + 2 + 1 = 6 = \frac{\mu}{2}$$

Another interpretation: patchworking of a Harnack curve



$v = \frac{\mu}{2} = 6$

Counterexamples to the existence of M-smoothings

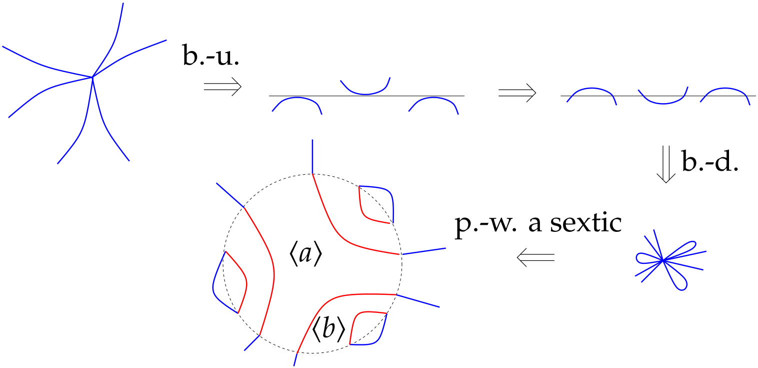
Another statement from [Risler, 1987] with a similar inductive blow-up construction:

Theorem

Let (C, z) have $r = r_{\mathbb{R}} \geq 2$ real branches that are tangent to r distinct lines. Then (C, z) admits an M-smoothing.

Observation: In some cases the proof **fails** even for $r = 2$ (Kharlamov, 1988)

Example: $x^2y^2(x-y)^2 + (x+y)^5(x^2 - 5xy + y^2) = 0$,
local Harnack bound $v \le \frac{\mu - r + 1}{2} = 13$



Fact: $a + b < 10$ (Orevkov, 1999)

Theorem (Kharlamov-Orevkov-Sh., 1999)

The three-cuspidal singularity

$$x^2y^2(x-y)^2 + (x+y)^5(x^2 - 5xy + y^2) = 0$$

has no M-smoothing.

Proof: Hard work with all available topological restrictions.

More on existence of maximal smoothings

We present results that are due to Chevallier (1997), Kharlamov-Risler (1995), Kharlamov-Risler-Sh. (2001), and they are obtained via a suitably modified blow-up construction and/or Viro's pathworking construction

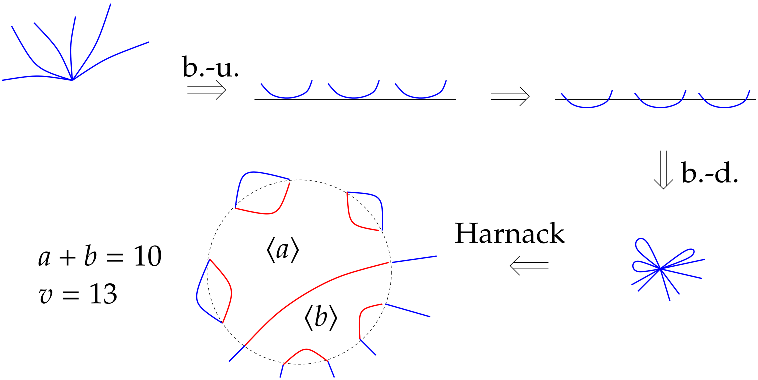
The most general result (i.e., concerning arbitrary real singularities) answers a question asked by S. Orevkov:

Theorem (Kharlamov-Risler-Shustin)

For any real plane curve singularity there exists a topologically equivalent over $\mathbb{C}$ real plane curve singularity that has only real branches and admits an M-smoothing

Proof. Blow-up construction

Example: Three-cuspidal singularity $x^2y^2(x-y)^2 + (x+y)^7 = 0$, local Harnack bound $v \leq \frac{\mu - r + 1}{2} = 13$



M-smoothings for specific classes of singularities

Theorem (Chevallier, Kharlamov-Risler-Sh.)

Any singularity whose branches all are real and smooth admits an M-smoothing.

Theorem (Chevallier, Kharlamov-Risler)

Any singularity whose branches are real, have even multiplicity, and are located in the same halfplane, admits an M-smoothing.

Both proofs. Blow-up construction

Theorem (Chevallier, Kharlamov-Risler-Sh.)

For any real Newton nondegenerate singularity there exists an equisingular equivariant deformation into a singular point that admits a weak M-smoothing.

Proof. Viro's patchworking construction

Semiquasihomogeneous singularities

A semiquasihomogemenous singularity is that of the form

$$x^a y^b \prod_{i=1}^{k} (y^p - \lambda_i x^q) + \text{h.o.t.} = 0$$

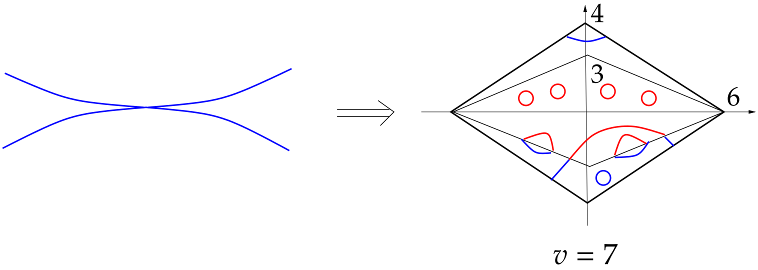
$$(p,q) = 1, \quad a,b \leq 1, \quad \lambda_i \neq \lambda_j \text{ for } i \neq j$$

Theorem (Kharlamov-Risler-Sh.)

(1) Any real SQH singular point, whose real roots λ_i (if any) have the same sign, admits an M-smoothing.
(2) For any real SQH singular point there exists an equisingular equivariant deformation into a real SQH singular point that admits an M-smoothing.

Proof. Viro's patchworking construction

Example: Two-cuspidal singularity $y^4 - x^6 = 0$,
local Harnack bound $v \leq \frac{\mu - r + 1}{2} = 7$



Note: Here $l = 1$ in the refined local Harnack inequality

Open problems

(1) *What are general geometric phenomena behind the lack of M-smoothing?*

(2) *Does any real SQH singularity admit an M-smoothing?*

(3) a) *Does any real curve singularity admit a weak M-smoothing?*
b) *Can any real curve singularity be deformed in an equisingular equivariant family into a singularity admitting a weak M-smoothing?*

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