Perspectives in Real Geometry
Conference in memory of Jean-Jacques Risler

September 18-22, 2017

Monday, September 18th

7h30 - 9h00	Breakfast
9h15 - 9h30	*Opening*
9h30 - 10h30	**Marie-Françoise Roy** (Université de Rennes 1) *Quantitative aspect of two algebraic proofs of the fundamental theorem of algebra*
10h30 - 11h00	Coffee
11h00 - 12h00	**Bernard Teissier** (CNRS) *Extensions of valuations to completion and henselization*
12h30 - 13h30	Lunch
16h30 - 17h30	**Benoît Bertrand** (Université de Toulouse III) *Total curvature of real algebraic and tropical curves and hypersurfaces*
17h30 - 18h00	Coffee
18h00 - 19h00	**Frédéric Mangolte** (Université d'Angers) *New results on fake real planes*
19h30 - 20h30	Dinner

Tuesday, September 19th

7h30 - 9h00	Breakfast
9h30 - 10h30	**Antonio Lerario** (SISSA) *Probabilistic enumerative geometry*
10h30 - 11h00	Coffee
11h00 - 12h00	**Jean-Yves Welschinger** (CNRS) *Expected topology of a random subcomplex in a simplicial complex*
12h30 - 13h30	Lunch
16h30 - 17h30	**Penka Georgieva** (Université Pierre et Marie Curie) *Real curves and a Klein TQFT*
17h30 - 18h00	Coffee
18h00 - 19h00	**Grigory Mikhalkin** (Université de Genève) *Menelaus-type enumerative problems and simple Harnack curves*
19h30 - 20h30	Dinner

Wednesday, September 20th

7h30 - 9h00	Breakfast
9h00 - 10h00	**Ugo Boscain** (CNRS) *Curvature and heat diffusion in almost-Riemannian geometry*
10h00 - 10h20	Coffee
10h20 - 11h20	**Yosef Yomdin** (Weizmann Institute) *Smooth parametrizations, their old and new applications, and possible connections to Singularity Theory*
11h30 - 12h30	**Stepan Orevkov** (Université de Toulouse III) *On real algebraic knots and links*
12h30 - 13h30	Lunch
	Free afternoon
19h30 - 20h30	Dinner
20h45 - 21h45	**Eugenii Shustin** (Tel Aviv University) *A local Harnack problem (after J.-J. Risler)*

Thursday, September 21st

7h30 - 9h00	Breakfast
9h30 - 10h30	**Olivier Benoist** (CNRS) *Sums of three squares and Noether-Lefschetz loci*
10h30 - 11h00	Coffee
11h00 - 12h00	**Mikhail Skopenkov** (NRU Higher school of economics, Institute for information transmission problems RAS) *Surfaces containing two circles through each point*
12h30 - 13h30	Lunch
16h30 - 17h30	**Askold Khovanskii** (University of Toronto) *Complex torus, its good compactifications and the ring of conditions*
17h30 - 18h00	Coffee
18h00 - 19h00	**Pedro Gonzáles Pérez** (Universidad Complutense) *On multi-Harnack smoothings of real plane branches*
19h30 - 21h00	Dinner

Friday, September 22nd

7h30 - 9h00	Breakfast
9h30 - 10h30	**Sergey Finashin** (Middle East Technical University) *Deformation of lines in Real Cubic Threefolds*
10h30 - 11h00	Coffee
11h00 - 12h00	**Alex Degtyarev** (Bilkent University) *Lines on smooth $K3$-surfaces*
12h30 - 13h30	Lunch
13h30 - 14h30	**Adam Parusinski** (Université de Nice) *Weight Filtration for Real Algebraic Varieties*
14h45 - 15h45	**Kristin Shaw** (MPIM Leipzig) *The separating semigroup of a real curve*
19h30 - 20h30	Dinner

Abstracts

Oliver Benoist (CNRS)

Sums of three squares and Noether-Lefschetz loci

It is a theorem of Hilbert that a real polynomial in two variables that is nonnegative is a sum of 4 squares of rational functions. Cassels, Ellison and Pfister have shown the existence of such polynomials that are not sums of 3 squares of rational functions. In this talk, we will prove that those polynomials that may be written as sums of 3 squares are dense in the set of nonnegative polynomials. The proof is Hodge-theoretic.

Benoît Bertrand (Université de Toulouse III)

Total curvature of real algebraic and tropical curves and hypersurfaces

Risler's Inequality states that the curvature of the complex part of an affine real algebraic hypersurface is greater or equal than the curvature of its real part up to a universal multiplicative constant. Similar inequalities hold for the logarithmic image and its tropical limit. For curves of higher codimension, analogous inequalities also hold in the algebraic, "logarithmic" and tropical cases. Jean-Jacques was interested in whether those inequalities were sharp.

I will discuss the work he did, with L. López de Medrano and me, which compares the tropical case to the classical and "logarithmic" one in both dimension one and codimension one cases.

Ugo Boscain (CNRS)

Curvature and heat diffusion in almost-Riemannian geometry

In this talk I will review certain analytical and geometrical results in 2D-almost-Riemannian geometry. Two-dimensional almost-Riemannian structures are generalized Riemannian structures on surfaces for which a local orthonormal frame is given by a Lie bracket generating pair of vector fields that can become collinear. On the singular set (i.e. where these vector fields become collinear) all Riemannian quantities explodes, but geodesics are still well defined and smooth. Generically the singular set is a smooth curve.

In this talk I will review certain geometric properties (normal forms, Gauss-Bonnet theorem) and I will discuss the problem of the Schroedinger and of the heat evolution. In particular I will discuss if the singular set acts of not as a barrier. Some open problems will be also presented.

Alex Degtyarev (Bilkent University)

Lines on smooth $K3$-surfaces

I will settle a conjecture (originally based on the classification of smooth models of singular $K3$-surfaces) on the maximal number of lines in a sextic surface in $\mathbb{C}P^4$ (42 lines) or octic surface in $\mathbb{C}P^5$ (36 lines; the same bound is sharp for triquadrics). As in the case of spatial quartics, a classification of sextics and octics with many lines is also obtained. For example, there are several families of triquadrics with 32 lines other than the classical Kummer family (constructed via quadratic line complexes).

I will also discuss other polarizations of $K3$-surfaces. The asymptotic maximum is 24 lines, all lines lying in the fibers of an elliptic pencil. This bound is sharp for infinitely many polarizations.

In the original three cases of quartics in $\mathbb{C}P^3$, sextics in $\mathbb{C}P^4$, and octics in $\mathbb{C}P^5$, the line maximizing polarized $K3$-surfaces are also discriminant minimizing singular $K3$-surfaces admitting a smooth model. (It was this observation, valid in several known cases, together with the classification of small smooth models that gave rise to the conjecture in the first place.) However, this tendency does not persist: it fails already for surfaces of degree ten in $\mathbb{C}P^6$.

The principal tools are the theory of periods of $K3$-surfaces and Nikulin's theory of discriminant forms.

This work is partially supported by TÜBITAK project 116F211.

Sergey Finashin (Middle East Technical University)

Deformation of lines in Real Cubic Threefolds

In 1950s B.Segre has classified the real lines on non-singular cubic surfaces and found the monodromy groups acting on the real lines of such surfaces. In a joint work with V.Kharlamov we achieved a similar goal for the lines on real cubic threefolds, X. More precisely, we described the monodromy action on the connected components of the Fano surfaces parameterizing the set of real lines on X. The monodromy orbits are related to real non-singular quintic curves that are discrimilant loci of the conic bundles associated with the corresponding central projections.

Penka Georgieva (Université Pierre et Marie Curie)

Real curves and a Klein TQFT

The local Gromov-Witten theory of curves studied by Bryan and Pandharipande revealed strong structural results for the local GW invariants, which were later used by Ionel and Parker in the proof of the Gopakumar-Vafa conjecture. In this talk I will report on a joint work in progress with Eleny Ionel on the extension of these results to the real setting.

Pedro Gonzáles Pérez (Universidad Complutense)

On multi-Harnack smoothings of real plane branches

In 1987 Risler proved that any real plane branch (C, O) has a M-smoothing, i.e., a smoothing with the maximal number of connected components. His technique was a generalization of the classical Harnack's construction of M-curves by small perturbations. In this talk I will present a previous work with Risler giving another construction of M-smoothings of real plane branches based on Viro's patchworking method. We introduced a class of multiparametrical smoothings called "multi-Harnack", which are constructed iteratively at each stage of the toric resolution process, starting by the last one; at each step an "Harnack smoothing" is built from the singularity obtained by blowing-down the previous one. A Harnack smoothing is an M-smoothing which is in maximal position with respect to the coordinate axes. The main result is that the topological type of a multi-Harnack smoothing is determined by the complex equisingularity type of the branch (C, O). This result can be seen as a local version of a theorem of Mikhalkin about Harnack curves on real toric surfaces.

Askold Khovanskii (University of Toronto)

Complex torus, its good compactifications and the ring of conditions

Let X be an algebraic subvariety in $(\mathbb{C}^*)^n$. According to the good compactification theorem there is a complete toric variety $M \supset (\mathbb{C}^*)^n$ such that the closure of X in M does not intersect orbits in M of codimension bigger than $\dim_{\mathbb{C}} X$. All proofs of this theorem I met in literature are rather involved.

The ring of conditions of $(\mathbb{C}^*)^n$ was introduced by De Concini and Procesi in 1980-th. It is a version of intersection theory for algebraic cycles in $(\mathbb{C}^*)^n$. Its construction is based on the good compactification theorem. Recently two nice geometric descriptions of this

ring were found. Tropical geometry provides the first description. The second one can be formulated in terms of volume function on the cone of convex polyhedra with integral vertices in $\mathbb{R}^n$. These descriptions are unified by the theory of toric varieties.
I am going to discuss these descriptions of the ring of conditions and to present a new version of the good compactification theorem. This version is stronger that the usual one and its proof is elementary.

Antonio Lerario (SISSA)
Probabilistic enumerative geometry
A classical problem in enumerative geometry is the count of the number of linear spaces satisfying some geometric conditions (e.g. the number of lines on a generic cubic surface, the number of lines meeting four generic lines in projective space...). These problems are usually approached with the technique of Schubert Calculus, which describes how cycles intersect in the Grassmannian.
In this talk I will present a novel, more analytical approach to these questions. This comes after adopting a probabilistic point of view, the main idea is the replacement of the word generic with random. Of course over the complex numbers this gives the same answer, but it also allows to compute other quantities especially meaningful over the reals, where the generic number of solutions is not defined (e.g. the signed count or the average count). (This is based on joint works with P. Bürgisser, with S. Basu, E. Lundberg and C. Peterson and with K. Kozhasov).

Frédéric Mangolte (Université d'Angers)
New results on fake real planes
We study topologically minimal complexifications of the Euclidean affine plane $\mathbb{R}^2$ up to isomorphism and up to birational diffeomorphism. A fake real plane is a smooth geometrically integral surface S defined over R such that:
- The real locus $S(\mathbb{R})$ is diffeomorphic to $\mathbb{R}^2$;
- The complex surface $S(\mathbb{C})$ has the rational homology type of $A^2_{\mathbb{C}}$.;
- S is not isomorphic to $A^2_{\mathbb{R}}$ as surfaces defined over $\mathbb{R}$.

We prove that fake real planes exist by giving many examples and we tackle the question: does there exist fake planes S such that$S(\mathbb{R})$ is not birationally diffeomorphic to $A^2_{\mathbb{R}}$?

Grigory Mikhalkin (Université de Genève)
Menelaus-type enumerative problems and simple Harnack curves
A complex toric surface X is a union of the torus $(\mathbb{C}^*)^2$ and a collection of toric divisors (called the boundary of X). A configuration P of points in the boundary of X is called a configuration of Menelaus (or Menelaus-Carnot) type if there exists a compact curve in X such that its intersection with the boundary of X is P.
For a generic real Menelaus-type configuration of points there is a finite number of rational curves in X with the property as above. However, it turns out, that a simple Harnack curve among them is unique. This enumerative property can be used to easily recover a 2006 theorem of Kenyon-Okounkov (recently generalized by Olarte) coordinatizing the space of simple Harnack curves.

Stepan Orevkov (Université de Toulouse III)
On real algebraic knots and links
I will present the following results on real algebraic spatial curves:

(1) (joint with Mikhalkin) Classification of smooth irreducible spatial real algebraic curves of genus 0 or 1 up to degree 6 up to rigid isotopy.
(2) (joint with Mikhalkin) Classification of smooth irreducible spatial real algebraic curves with maximal encomplexed writhe up to (not rigid yet) isotopy.
(3) Classification of smooth spatial real algebraic curves of genus 0 with two irreducible components up to degree 6 up to rigid isotopy, in particular, the first (as far as I know) example of two spatial real algebraic curves which are isotopic, have equal degree, genus and encomplexed writhe of each irreducible component but not rigidly isotopic.

Adam Parusinski (Université de Nice)

Weight Filtration for Real Algebraic Varieties

We present an overview of the theory of weight filtration for real algebraic varieties. Recall that for real algebraic varieties we constructed functorial weight filtrations on homologies (or cohomologies) with Z/2 coefficients. These filtrations are analogs of Deligne's weight filtrations for complex algebraic varieties and can be defined both on classical homologies and on Borel-Moore homologies. We discuss the applications and open problems. (Based on a joint work with Clint McCrory)

Mikhail Skopenkov (NRU Higher school of economics, Institute for information transmission problems RAS)

Surfaces containing two circles through each point

Motivated by potential applications in architecture, we find all analytic surfaces in 3-dimensional Euclidean space such that through each point of the surface one can draw two transversal circular arcs fully contained in the surface. The search for such surfaces traces back to the works of Darboux from XIXth century. We prove that such a surface is an image of a subset of one of the following sets under some composition of inversions:

- the set $\{p + q : p \in P, q \in Q\}$, where P and Q are two circles in 3-dimensional Euclidean space;
- the set $\{2[pxq]/|p + q|^2 : p \in P, q \in Q\}$, where P and Q are two circles in the unit 2-dimensional sphere;
- the set $\{(x, y, z) : A(x, y, z, x^2 + y^2 + z^2) = 0\}$, where A is a polynomial in $\mathbb{R}[x, y, z, t]$ of degree 2 or 1.

The proof uses a new factorization technique for quaternionic polynomials. A substantial part of the talk is elementary and is accessible for high school students. The research is supported in part by President of the Russian Federation grant MK-6137.2016.1.
This is a joint work R. Krasauskas.

Kristin Shaw (MPIM Leipzig)

The separating semigroup of a real curve

This talk is based on joint work with Mario Kummer where we introduce the separating semigroup of a real algebraic curve of dividing type. The elements of this semigroup record the possible degrees of the covering maps obtained by restricting separating morphisms to the real part of the curve. We also introduce the hyperbolic semigroup which consists of elements of the separating semigroup arising from morphisms which are compositions of a linear projection with an embedding of the curve to some projective space. We completely determine both semigroups in the case of maximal curves. We also prove that any embedding of a real curve to projective space of sufficiently high degree is hyperbolic. Using these semigroups we show that the hyperbolicity locus of an embedded curve is in general not connected.

Eugenii Shustin (Tel Aviv University)
A local Harnack problem (after J.-J. Risler)
We give a historical overview of the local Harnack problem starting with the seminal J.-J. Risler's Inventiones paper in 1987 and presenting further activities and questions related to this topic.

Marie-Françoise Roy (Université de Rennes 1)
Quantitative aspect of two algebraic proofs of the fundamental theorem of algebra
The best known algebraic proof of the fundamental theorem of algebra is due to Laplace and uses the intermediate value axiom for polynomials of degree about d^d in the proof of existence of the complex roots of a polynomial of degree d. Much more recently Michael Eisermann proposed another algebraic proof using Cauchy index and winding numbers, his initial proof did not provide bounds on the degree of the intermediate value axiom needed in the proof of existence of the complex roots of a polynomial of degree d. Using subresultants, we obtain a modification of his proof, using the intermediate value axiom for polynomials of degree about d^2 in the proof of existence of the complex roots of a polynomial of degree d. This illustrates a quantitative difference between these two proofs. Work in progress with Daniel Perrucci.

Bernard Teissier (CNRS)
Extensions of valuations to completion and henselization
I will explain the motivations for the study of such extensions, which come essentially from the study of spaces of valuations and from trying to make in the algebraic world moves for resolution of singularities which we know how to make in the formal world.
I will concentrate on a proof of the uniqueness of the extension of valuations from a local domain to its henselization, which is based on a very down to earth approach to henselization through the Newton-Hensel method for approximating roots of a polynomial. (Joint work with Ana Belén de Felipe.)

Jean-Yves Welschinger (CNRS)
Expected topology of a random subcomplex in a simplicial complex
I will explain how to bound from above and below the expected Betti numbers of a random subcomplex in a simplicial complex and get asymptotic results under infinitely many barycentric subdivisions. This is a joint work with Nermin Salepci. It complements previous joint works with Damien Gayet on random topology.

Yosef Yomdin (Weizmann Institute)
Smooth parametrizations, their old and new applications, and possible connections to Singularity Theory
Smooth parametrization consists in a subdivision of a mathematical object under consideration into simple pieces, and then parametric representation of each piece, while keeping control of high order derivatives. Main examples for this talk are C^k or analytic parametrizations of semi-algebraic and o-minimal sets.
We provide an overview of some results and open problems on smooth parametrizations and their applications in several apparently rather separated domains: Smooth Dynamics, Diophantine Geometry, and Analysis. The structure of the results, open problems,

and conjectures in each of these domains shows in many cases a remarkable similarity, which we plan to stress.
We plan to discuss some apparent connections to Singularity Theory (in particular, topology and geometry of vanishing cycles).