A Complicial Compendium

Dominic Verity
Centre of Australian Category Theory,
Macquarie University.

27^{th} Sept 2017, Cats in HT&R
CIRM, Luminy

The WHY of Complicial Sets

Simplicial sets are lovely objects about which algebraic topologists know a lot. If something is described as a simplicial set, it is ready to be absorbed into topology. Or, in other words, no matter which definition of weak ω-category eventually becomes dominant, it will be valuable to know its simplicial nerve.

Ross Street (1987)
Algebra of Oriented Simplices

What is a Complicial Set?

Nerves of (strict) ω-Categories

Marked Simplicial Sets

Kan-type Fillers for Admissible Horns

Simplices as Strict ω-Categories

$$\mathcal{O} : \Delta \longrightarrow \omega\text{-}\underline{\mathrm{Cat}}$$

Orientals Functor

$$\mathcal{O}^0 := \boxed{0}$$

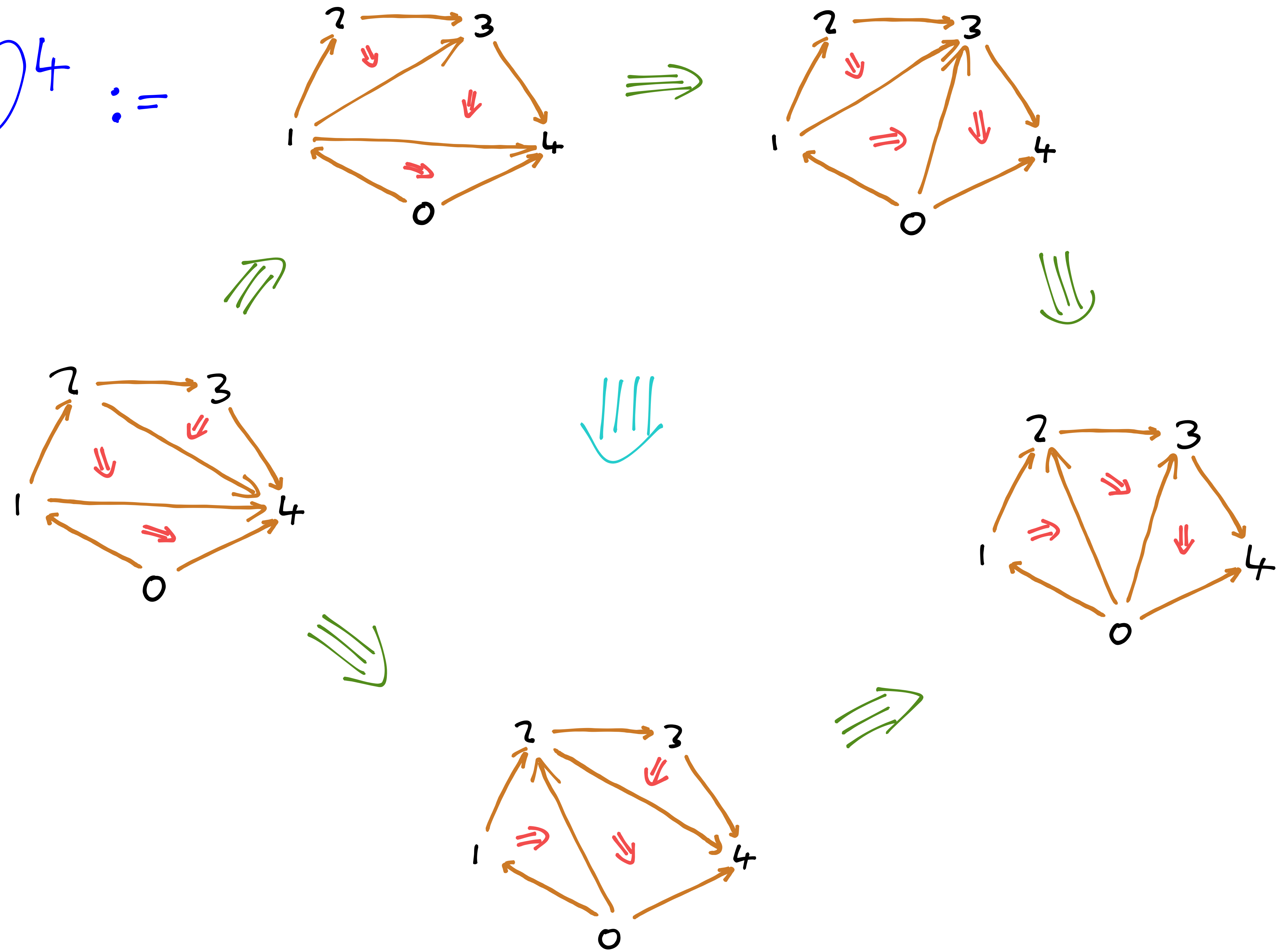
$$\mathcal{O}^1 := \begin{tikzcd} 0 \arrow[r] & 1 \end{tikzcd}$$

$$\mathcal{O}^2 := \begin{tikzcd} & 1 \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ 0 \arrow[ur] \arrow[rr] & {} & 2 \end{tikzcd}$$

$$\mathcal{O}^3 := \begin{tikzcd} & 1 \arrow[r] \arrow[drr] \arrow[d, phantom, "\Downarrow"] & 2 \arrow[dr] \arrow[d, phantom, "\Swarrow"] & \\ 0 \arrow[ur] \arrow[rrr] & {} & {} & 3 \end{tikzcd} \Rrightarrow \begin{tikzcd} & 1 \arrow[r] \arrow[d, phantom, "\Downarrow"] & 2 \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ 0 \arrow[ur] \arrow[urr] \arrow[rrr] & {} & {} & 3 \end{tikzcd}$$

The Maclane Pentagon

$\mathcal{O}^4 :=$



The Street Nerve

$$\begin{tikzcd}
\omega\text{-}\underline{\mathcal{C}at} \arrow[rr, bend right, "N_s"'] \arrow[rr, phantom, "\perp"] & & \underline{s\mathcal{S}et} \arrow[ll, bend right, "R_s"']
\end{tikzcd}$$

ω-CATEGORICAL REALISATION

APPLY KAN'S CONSTRUCTION TO THE FUNCTOR $\mathcal{O}$

$$N_s(\mathcal{C})_n := \mathrm{Hom}_{\omega\text{-}\mathcal{C}at}(\mathcal{O}^n, \mathcal{C})$$

QUESTIONS

OUR EXPERIENCE OF THE CLASSICAL NERVE CONSTRUCTION FOR CATEGORIES LEADS US TO WONDER:

1) IS STREET'S NERVE FULLY FAITHFUL?

SPOILER: THIS IS ONLY ALMOST TRUE.

2) HOW CAN WE CHARACTERISE ITS ESSENTIAL IMAGE?

Marked Simplicial Sets

(Roberts 1977)

A marked simplicial set consists of:

- •) A simplicial set X accompanied by
- •) A subset $mX \subseteq X$ of simplices of arbitrary dimension

Elements of this subset are said to be marked

subject to the stipulation that

- •) Every degenerate simplex of X is a member of mX.

A map $f:(X, mX) \longrightarrow (Y, mY)$ of marked simplicial sets is a simplicial map $f: X \longrightarrow Y$ that preserves marks.

$f(mX) \subseteq mY$

$\underline{msSet}$:= category (quasi-topos) of marked simplicial sets and mark preserving simplicial maps

MARKED NERVES

A SIMPLEX $x : \mathcal{O}^n \longrightarrow \mathcal{C}$ OF THE NERVE $N_s(\mathcal{C})$ IS SAID TO BE THIN IF IT MAPS THE UNIQUE NON-IDENTITY n-CELL OF $\mathcal{O}^n$ TO AN IDENTITY n-CELL IN $\mathcal{C}$.

THE SET OF THIN SIMPLICES IS A MARKING OF THE NERVE $N_s(\mathcal{C})$ CALLED THE ROBERTS MARKING.

THIS EXTENDS STREET'S NERVE TO AN ADJUNCTION:

$$\begin{tikzcd} \underline{\omega\text{-}\mathcal{C}at} \arrow[r, bend right, "N_s"'] \arrow[r, phantom, "\perp"] & \underline{msSet} \arrow[l, bend right, "R_s"'] \end{tikzcd}$$

A Keystone Result

The Street nerve functor $N_s : \underline{\omega\text{-Cat}} \longrightarrow \underline{\text{msSet}}$ is full + faithful.

•) Conjectured by Street (1987).

•) Proved by Verity (2007-ish).

Roberts characterisation
(explained later)

Proof relies upon:

(i) Kan-like horn filler properties of Street nerves.

(ii) The introduction of a Gray tensor product of marked simplicial sets.

(iii) A path category construction for marked simplicial sets that satisfy the conditions in (i).

(iv) An ordinal subdivision analysis of ω-categorical realisations of Gray tensors of simplices.

LAX GRAY TENSOR

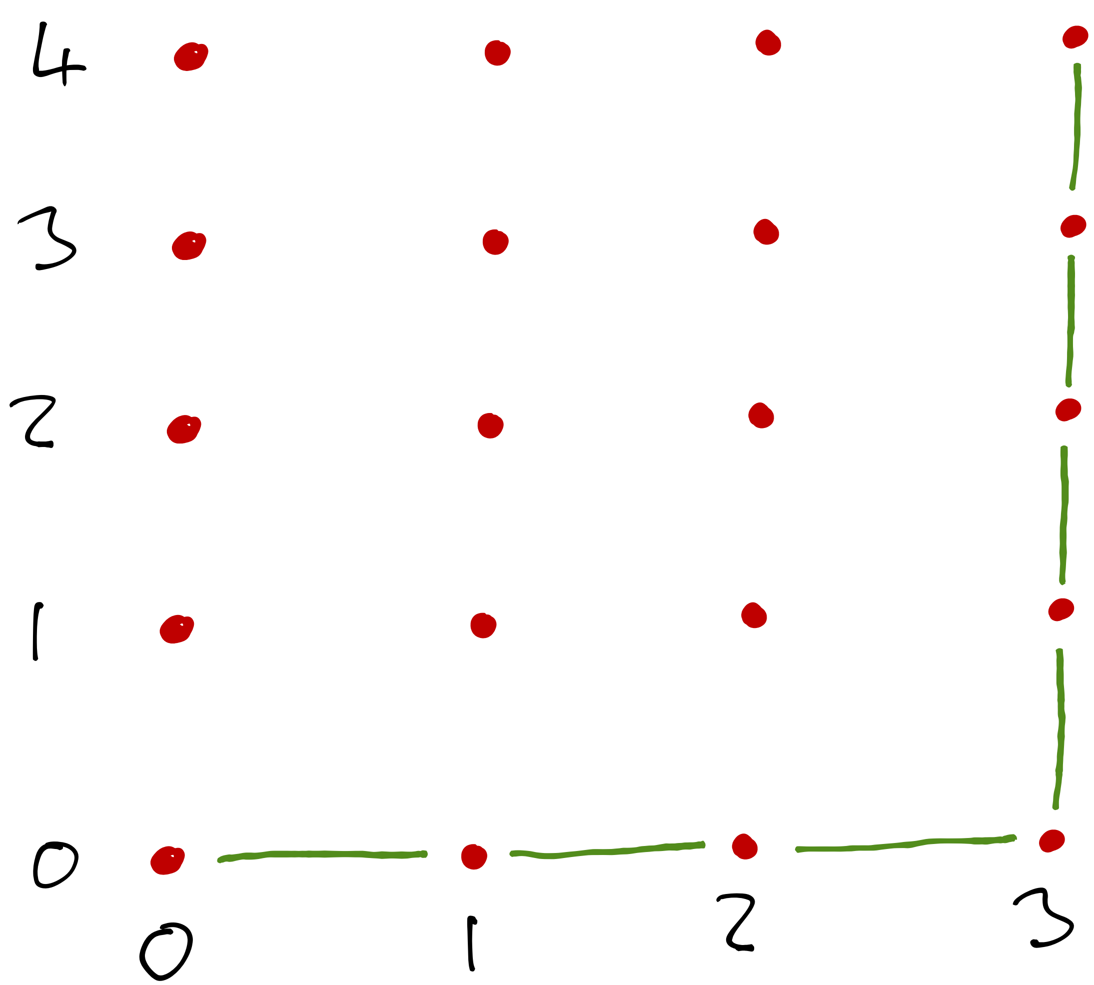
X AND Y MARKED SIMPLICIAL SETS.

$X \otimes Y$ THE LAX GRAY TENSOR HAS

- UNDERLYING SIMPLICIAL SET $X \times Y$,
- n-SIMPLEX (x, y) MARKED IF FOR ALL $i + j = n$ EITHER $x \cdot \amalg^0_{i,j}$ MARKED IN X OR $y \cdot \amalg^1_{i,j}$ MARKED IN Y.

$\amalg^0_{i,j} : [i] \to [n]$, $k \mapsto k$

$\amalg^1_{i,j} : [j] \to [n]$, $k \mapsto k+i$



PSEUDO-GRAY = PRODUCT.

$\Delta^3 \otimes \Delta^4$

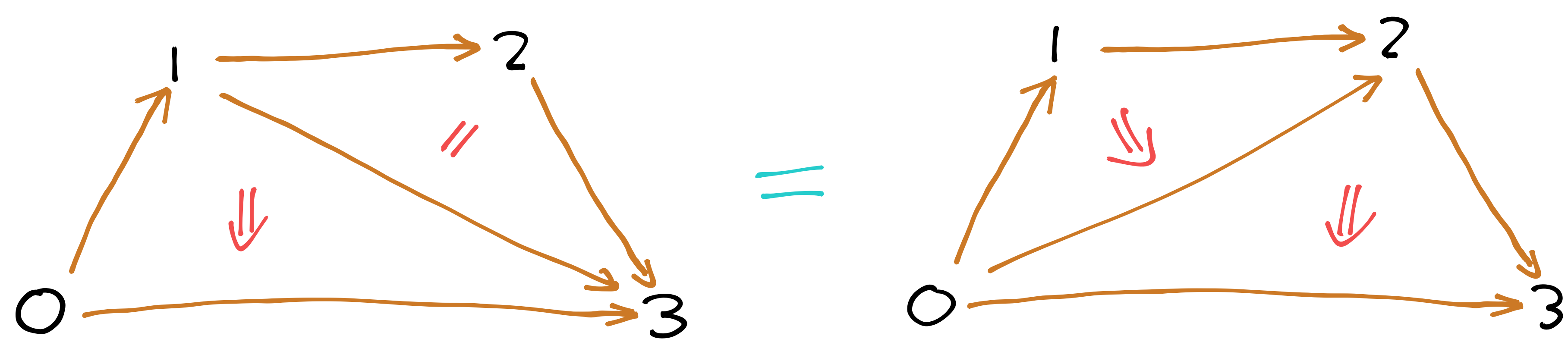
ADMISSIBLE SIMPLICES

- Δ^n THE USUAL STANDARD SIMPLEX IN WHICH ONLY THE DEGENERATE SIMPLICES ARE MARKED.

- $\Delta^{n,k}$ (FOR $k=0,\dots,n$) CONSTRUCTED FROM Δ^n BY ALSO MARKING ALL FACES THAT HAVE THE INTEGERS $\{k-1,k,k+1\}\cap[n]$ AS VERTICES.

THE k^{th} ADMISSIBLE n-SIMPLEX

$R_s(\Delta^{3,2}) =$



- $\Lambda^{n,k}$ (FOR $k=0,\dots,n$) THE USUAL (n,k)-HORN WITH MARKING INHERITED FROM $\Delta^{n,k}$.

SLOGAN THE INNER ADMISSIBLE SIMPLEX $\Delta^{n,k}$ DESCRIBES HOW ITS $(n-1)$-FACE δ^k MAY BE OBTAINED AS A COMPOSITE OF ITS $(n-1)$-FACES $\delta^{k-1}, \delta^{k+1}$ ALONG THEIR COMMON $(n-2)$-FACE

$$\begin{tikzcd} & 1 \arrow[r, "g"] \arrow[drr, "hg"'] \arrow[d, phantom, "\Downarrow \beta\cdot h\alpha"] & 2 \arrow[dr, "h"] \arrow[d, phantom, "="] & \\ 0 \arrow[ur, "f"] \arrow[rrr, "\iota"'] & {} & {} & 3 \end{tikzcd} \quad = \quad \begin{tikzcd} & 1 \arrow[r, "g"] \arrow[d, phantom, "\Downarrow \alpha"] & 2 \arrow[dr, "h"] \arrow[d, phantom, "\Downarrow \beta"] & \\ 0 \arrow[ur, "f"] \arrow[urr, "k"] \arrow[rrr, "\iota"'] & {} & {} & 3 \end{tikzcd}$$

SOME OTHER MARKED SIMPLICES

- $\#\Delta^n$ CONSTRUCTED FROM THE STANDARD n-SIMPLEX Δ^n BY MARKING ITS UNIQUE NON-DEGENERATE n-DIMENSIONAL FACE $id_n : [n] \longrightarrow [n]$.
- $\natural\Delta^{n,k}$ CONSTRUCTED FROM THE ADMISSIBLE n-SIMPLEX $\Delta^{n,k}$ BY MARKING ITS $(n-1)$-DIMENSIONAL FACES $\delta_n^i : [n-1] \longrightarrow [n]$ FOR $i \in \{k-1, k+1\} \cap [n]$.
- $\#\Delta^{n,k}$ CONSTRUCTED FROM THE n-SIMPLEX $\natural\Delta^{n,k}$ BY ALSO MARKING THE $(n-1)$-DIMENSIONAL FACE $\delta_n^k : [n-1] \longrightarrow [n]$

THE ROBERTS CHARACTERISATION

(PRELUDE TO COMPLICIAL SETS)

SUPPOSE THAT A IS A MARKED SIMPLICIAL SET, WE SAY THAT IT IS:

PRE-COMPLICIAL $\equiv$

RIGHT LIFTING PROPERTY

$$\begin{tikzcd} {}^{\natural}\Lambda^{n,k} \arrow[rr, "a"] \arrow[d, hook] & & A \\ {}^{\#}\Lambda^{n,k} \arrow[urr, dashed, "\exists!"'] & & \end{tikzcd}$$

$(\forall\, n \geq 2,\ k = 0, \ldots, n)$

"MARKED SIMPLICES ARE CLOSED UNDER SIMPLICIAL COMPOSITION"

STRICTLY COMPLICIAL $\equiv$

PRE-COMPLICIAL + UNIQUE RLP

$$\begin{tikzcd} \Lambda^{n,k} \arrow[rr, "a"] \arrow[d, hook] & & A \\ \Lambda^{n,k} \arrow[urr, dashed, "\exists!"'] & & \end{tikzcd}$$

$(\forall\, n \geq 1,\ k = 0, \ldots, n)$

"ALL COMPOSABLE SIMPLICES HAVE A SIMPLICIAL COMPOSITE"

+ ALL MARKED 1-SIMPLICES ARE DEGENERATE.

COMPLICIAL SETS

STREET'S INSIGHT (1987): BY WEAKENING THE LIFTING PROPERTY IN ROBERTS' CHARACTERISATION WE MIGHT HOPE TO CONSTRUCT A MODEL OF (∞,∞)-CATEGORIES!

~~STRICTLY~~ COMPLICIAL $\equiv$

PRE-COMPLICIAL + ~~UNIQUE~~ RLP

$$\begin{tikzcd} \Lambda^{n,k} \arrow[r, "a"] \arrow[d, hook] & A \\ \Delta^{n,k} \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

"ALL COMPOSABLE SIMPLICES HAVE A WEAK SIMPLICIAL COMPOSITE"

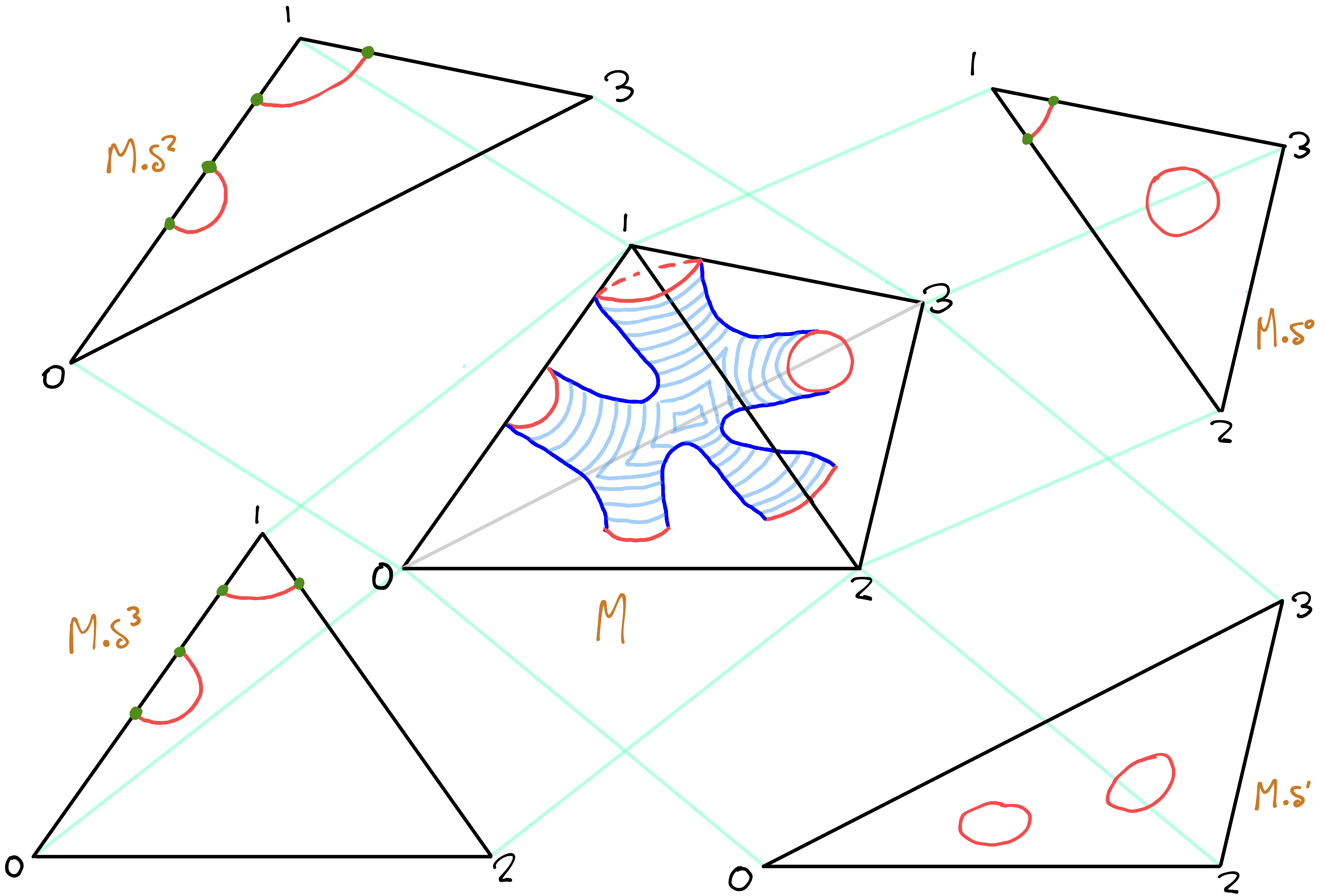
$(\forall\, n \geq 1,\ k = 0, \ldots, n)$

+ ~~ALL MARKED 1-SIMPLICES ARE DEGENERATE.~~

NOW WE THINK OF MARKED SIMPLICES AS BEING EQUIVALENCES RATHER THAN IDENTITIES

SIMPLICIAL BORDISMS

(AN EXAMPLE)



SIMPLICIAL BORDISMS (MORE FORMALLY)

A SIMPLICIAL BORDISM IS A FUNCTOR:

$$\Delta_f \downarrow [n] \xrightarrow{M} \underline{\text{PL-MAN}}_e$$

SUBCATEGORY OF FACE MAPS IN Δ

(PL) MANIFOLDS AND REGULAR EMBEDDINGS

SUBJECT TO THE CONDITIONS THAT:

1) FOR EACH OBJECT $\sigma : [r] \to [n]$ OF $\Delta_f \downarrow [n]$ WE HAVE THAT FORALL $i \in [r]$ EITHER:
   - $M(\sigma\delta^i) = \emptyset$ OR
   - $\dim(M(\sigma\delta^i)) = \dim(M(\sigma)) - 1$ AND $M(\sigma\delta^i) \subseteq \partial M(\sigma)$.

2) FOR EACH $x \in M$ THERE EXISTS A UNIQUE FACE MAP $\sigma : [r] \longrightarrow [n]$ SUCH THAT $x \in \text{interior}(M(\sigma))$.

THE KAN COMPLEX OF BORDISMS

LET $\mathcal{B}_d$ DENOTE THE SEMI-SIMPLICIAL SET WITH

- n-SIMPLICES THE SIMPLICIAL BORDISMS $M : \Delta_s[n] \longrightarrow \underline{\mathrm{PL}\text{-}\mathrm{Man}_e}$
- ACTION OF A FACE OPERATOR $\sigma : [r] \longrightarrow [n]$ DEFINED TO CARRY A SIMPLICIAL BORDISM M TO $M \cdot \sigma$ GIVEN BY:

$$(M \cdot \sigma)(\beta) := M(\sigma\beta)$$

PROP THE SEMI-SIMPLICIAL SET $\mathcal{B}_d$ ADMITS FILLERS FOR ALL HORNS:

$$\begin{tikzcd}
\Lambda^{n,k} \arrow[r, "h"] \arrow[d, hook] & \mathcal{B}_d \\
\Delta^n \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

SO WE MAY CONSTRUCT ACTIONS OF DEGENERACY OPERATORS WHICH MAKE IT INTO A GENUINE KAN COMPLEX.

A Proof Sketch

"Gluing"

N, M n-manifolds, P $(n-1)$-manifold

$$\begin{tikzcd} P \arrow[r, hook] \arrow[d, hook] & N \arrow[d, hook] \\ M \arrow[r, hook] & M \cup_P N \end{tikzcd}$$

If $P \hookrightarrow N$ and $P \hookrightarrow M$ are embeddings into ∂M and ∂N then $M \cup_P N$ is an n-manifold.

We may apply this result inductively to show that the $(n-1)$-faces of a horn $h : \Lambda^{n,k} \to B_d$ can be glued to give a simplicial bordism whose boundary coincides with that of the missing face of that horn.

"Stitching in"

Suppose that H is the manifold constructed by gluing together the $(n-1)$-faces of the horn $h : \Lambda^{n,k} \to B_d$.

We may attach the cylindrical manifold $H \times [0,1]$ to the horn by its end $H \times \{0\}$. This gives us an n-simplex $H : \Delta^n \to B_d$ whose restriction along $\Lambda^{n,k} \hookrightarrow \Delta^n$ is the original horn.

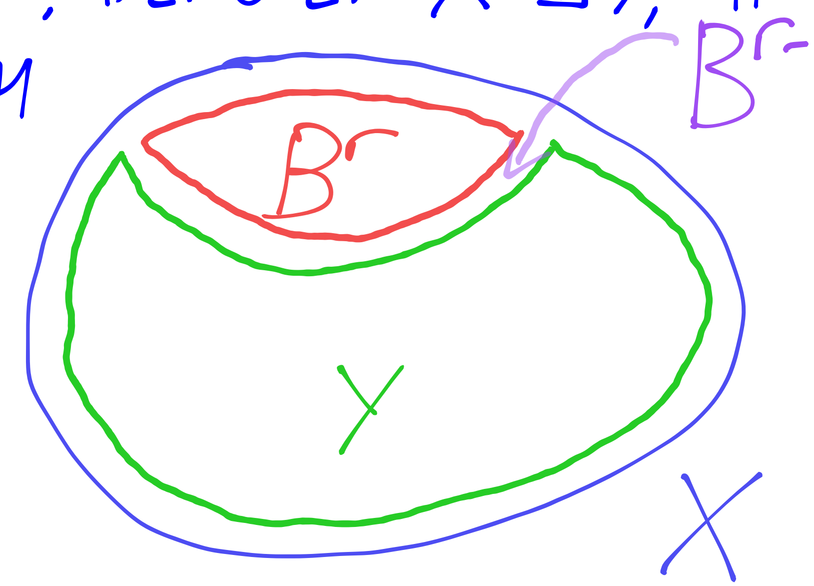
Its face $H \cdot \delta^k$ is $(H \times \{1\}) \cup (\partial H \times I)$. This is isomorphic to H by the collaring theorem for PL-manifolds.

COLLAPSING

AS A KAN COMPLEX B_d CLASSIFIES BORDISMS UPTO BORDISM. WE'D RATHER LIKE IT TO CLASSIFY BORDISMS UPTO TRIVIAL BORDISM. WE SHALL USE OUR MARKING TECHNOLOGY TO KEEP TRACK OF THE TRIVIAL BORDISMS.

<u>DEFN</u> IF X AND Y ARE POLYTOPES THEN THERE IS AN ELEMENTARY COLLAPSE FROM X TO Y, DENOTED $X \searrow Y$, IF IF THERE IS A BALL B^r IN X WITH

- $Y \cap B^r$ IS A FACE B^{r-1} OF ∂B^r
- $X = Y \cup B^r$



AND X COLLAPSES TO Y, WRITTEN $X \searrow Y$, IF $X \searrow X_1 \searrow X_2 \cdots\cdots \searrow Y$.

IF M IS A SIMPLICIAL BORDISM WE DEFINE

$$\partial^- M = \bigcup_{i \text{ EVEN}} M(\delta^i) \qquad \partial^- M = \bigcup_{i \text{ ODD}} M(\delta^i)$$

WE SHALL MARK M IN B_d IF THERE IS A COLLAPSE $M \searrow \partial^- M$.

B_α IS A COMPLICIAL SET

OR MORE PRECISELY B_α WITH THE MARKING DEFINED ABOVE IS A COMPCICIAL SET.

COMMENTS ABOUT THE PROOF OF THIS FACT:

1) WE MAY APPLY SHELLING THEORY TO SHOW THAT IF M IS A MARKED SIMPLEX IN B_α THEN $\partial^- M \cong \partial^+ M$ AND $M \cong \partial^- M \times [0,1]$. IN PARTICULAR $M \searrow \partial^- M$ IFF $M \searrow \partial^+ M$.

2) TO SHOW THAT B_α HAS FILLERS FOR ADMISSIBLE HORNS, IT IS ENOUGH TO SHOW THAT THE SPECIFIC FILLER THAT WE CONSTRUCTED ABOVE IS MARKED.

THIS FILLER HAS THE PROPERTY THAT $M \searrow \bigcup_{i \neq k} M(\delta_i)$
BUT A SIMPCE ARGUMENT, USING THE ADMISSIBILITY OF THE ORIGINAL HORN, SUFFICES TO EXTEND THAT TO A COLLAPSE $M \searrow \partial^- M$ IF k IS ODD OR $M \searrow \partial^+ M$ IF k IS EVEN

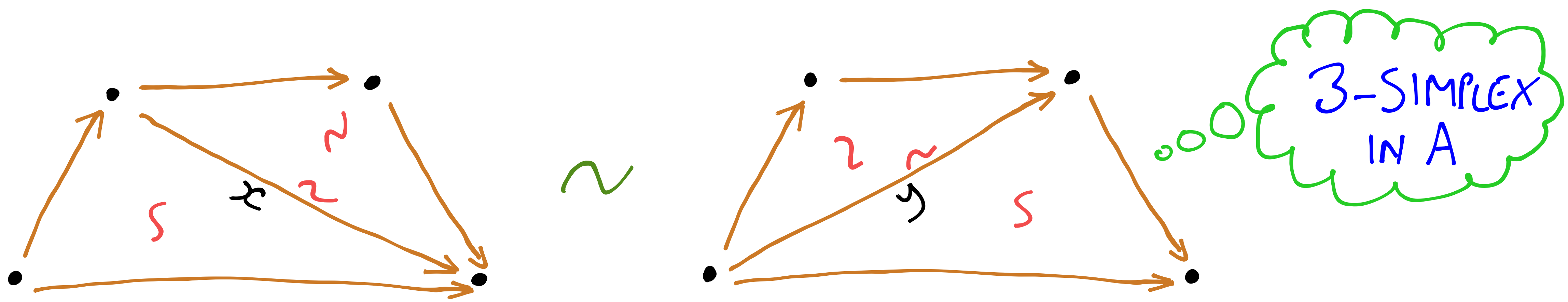
SATURATION

QUESTION DOES B_a HAVE ANY SIMPLICES THAT ARE MORALLY EQUIVALENCES BUT ARE NOT MARKED?

ANSWER YES, SINCE IT IS KNOWN THAT A COBORDISM CAN BE INVERTIBLE WITHOUT BEING TRIVIAL.

RIDER CAN WE DESCRIBE THE MORAL EQUIVALENCES AND MARK THEM WITHOUT DISRUPTING COMPLICIALITY?

DEFN A PRE-COMPLICIAL SET A IS SATURATED IF BOTH IT AND ITS SLICES SATISFY THE 2-OF-6 PROPERTY.



FACES OF DIMENSION ≥ 2 + 1-SIMPLICES x AND y MARKED

$\Rightarrow$ ALL 1-SIMPLICES ARE MARKED.

THE SATURATION OF B_d

DEFN THE SATURATION OF A PRE-COMPLICIAL SET IS THE SMALLEST EXPANSION OF ITS MARKING THAT MAKES IT BOTH SATURATED AND PRE-COMPLICIAL.

THEOREM THE SATURATION OF A COMPLICIAL SET IS AGAIN COMPLICIAL.

SLOGAN SATURATED COMPLICIAL SETS ARE A MODEL OF (∞,∞)-CATEGORIES. THE SATURATED B_d DESERVES TO BE THOUGHT OF AS BEING AN (∞,∞)-CATEGORY OF BORDISMS.

QUESTION IS THE SATURATED MARKING OF B_d DISTINCT FROM THE "ALL SIMPLICES" MARKING OF B_d AS A KAN COMPLEX?

ANSWER YES SINCE WE CAN SHOW THAT ANY MARKED SIMPLEX M IN SATURATED B_d IS AN h-COBORDISM FROM $\partial^- M \to \partial^+ M$

CURATED PLOT POINTS IN COMPLICIAL THEORY

1) LAX GRAY TENSOR OF (PRE-) COMPLICIAL SETS.

2) MODEL STRUCTURE WITH (SATURATED) COMPLICIAL SETS AS FIBRANT OBJECTS.

3) HOMOTOPY COHERENT NERVE

$$\begin{tikzcd} \underline{msSet} \arrow[rr, bend right, "N"'] \arrow[rr, phantom, "\perp"] & & \underline{msSet}\text{-}\underline{Cat} \arrow[ll, bend right, "\mathfrak{C}"'] \end{tikzcd}$$

QUILLEN WRT COMPLICIAL MODEL STRUCTURES.

4) COMPLETE SEGAL SPACES OF WEAK GLOBES

$$\begin{tikzcd} \underline{MSSet} \arrow[rr, bend right, "P"'] \arrow[rr, phantom, "\perp"] & & {[\Delta^{op}, \underline{MSSet}]} \arrow[ll, bend right, "R"'] \end{tikzcd}$$

QUILLEN WRT COMPLICIAL MS ON LEFT, COMPLETE SEGAL MS ON RIGHT.

CASE STUDY: COMPREHENSIVE FIBRATIONS

COMPLICIAL ANALOGS OF GROTHENDIECK FIBRATIONS.

(i) AUGMENT OUR PRE-COMPLICIAL SETS WITH A SECOND MARKING, USE THIS TO TRACK THOSE SIMPLICES THAT SHOULD BE REGARDED AS BEING (CO) CARTESIAN

(ii) DEFINE A COMPREHENSIVE (CO) FIBRATION TO ROUGHLY BE A MAP $p: E \twoheadrightarrow B$ OF AUGMENTED COMPLICIAL SETS WHICH

- IS A FIBRATION IN THE COMPLICIAL MODEL STRUCTURE AND
- PRESERVES AUXILIARY MARKINGS AND
- ADMITS LIFTS OF (LEFT) RIGHT OUTER HORNS WHICH ARE ADMISSIBLE WRT AUXILIARY MARKINGS

EXAMPLE GIVEN A 0-SIMPLEX $a \in A$ IN A COMPLICIAL SET, THE LAX (SIMPLICIAL) SLICE A/a IS AGAIN A COMPLICIAL SET, AND IT HAS A CANONICAL AUXILIARY MARKING THAT MAKES $p_a: A/a \twoheadrightarrow A$ INTO A COMPREHENSIVE FIBRATION.

THE CONTRAVARIANT REPRESENTABLE ON a

THE COMPREHENSION THEOREM

THM IF $\mathcal{E}$ AND $\mathcal{B}$ ARE ENRICHED IN COMPLICIAL SETS WITH AUXILIARY MARKING AND $F : \mathcal{E} \to \mathcal{B}$ IS AN ENRICHED FUNCTOR SUCH THAT

- EACH $F : \mathrm{Fun}_{\mathcal{E}}(A,B) \longrightarrow \mathrm{Fun}_{\mathcal{B}}(FA,FB)$ IS A COMPREHENSIVE COFIBRATION
- FOR ALL $B \in \mathcal{E}$, $f : X \longrightarrow FB \in \mathcal{B}$ THERE EXISTS A CARTESIAN LIFT $\chi_f : A \to B$ IN $\mathcal{E}$.

THEN THE HOMOTOPY COHERENT NERVE $N F : N\mathcal{E} \longrightarrow N\mathcal{B}$ IS A COMPREHENSIVE FIBRATION.

EXAMPLE $\underline{\mathcal{C}omp}$:= COMPLICIAL SETS WITH AUXILIARY MARKING

$\underline{\mathcal{C}om\mathcal{C}of}$:= (SMALL) COMPREHENSIVE COFIBRATIONS

A LIFTING ARGUMENT IN $\mathrm{cod} : N(\underline{\mathcal{C}om\mathcal{C}of}) \longrightarrow N(\underline{\mathcal{C}omp})$ SHOWS THAT ANY COMPREHENSIVE COFIBRATION $p : E \twoheadrightarrow B \in \underline{\mathcal{C}om\mathcal{C}of}$ GIVES RISE TO A FUNCTOR $c_p : B \longrightarrow N(\underline{\mathcal{C}omp})$ WHICH MAPS EACH $b \in B_0$ TO THE CORRESPONDING FIBRE E_b OF p.