A syntactic approach to polynomial functors, polynomial monads and opetopes

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Polynomial functors (a fibered presentation)

A polynomial functor P from a set of colors I to a set of colors J (notation $P : I \to J$) is given by the following data:

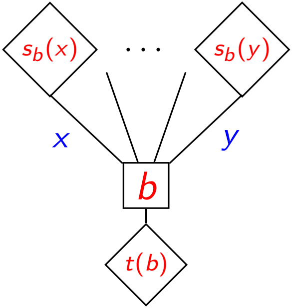
- A set B^P of operations and a mapping $\mathtt{t}^P : B^P \to J$ (the target color map),
- and for each $b \in B$, a (finite) set $\mathtt{A}^P(b)$ (the arity and a map $\mathtt{s}^P_b : \mathtt{A}^P(b) \to I$ (the source colors map).

These data can be recorded as **typing judgements**:

$$(\mathtt{A}^P(b), \mathtt{s}^P_b) \vdash^P b : \mathtt{t}^P(b)$$

Think of $(\mathtt{A}^P(b), \mathtt{s}^P_b)$ as a set of declarations $x : \mathtt{s}^P_b(x)$ $(x \in \mathtt{A}^P(b))$.

Polynomial functor (pictorially)



Note the difference between names and decorations: the latter can be repeated, while the former ones are in bijection with the number of wires going into the operation.

Polynomial functors (standard presentation)

Polynomial functors were introduced by Gambino and Kock (following ideas of Joyal) (2013), as a triple of maps

$$I \xleftarrow{\mathtt{s}} \mathtt{A} \xrightarrow{\mathtt{p}} B \xrightarrow{\mathtt{t}} J$$

We recover a polynomial functor in standard presentation from our presentation by taking

$$\mathtt{A} = \Sigma_{b \in B} \mathtt{A}(b) \qquad \mathtt{s}(b, x) = \mathtt{s}_b(x) \qquad \mathtt{p}(x, b) = b$$

Conversely, take $\mathtt{A}(b) = \mathtt{p}^{-1}(b)$ and $\mathtt{s}_b = \mathtt{s}_{\restriction_{\mathtt{A}(b)}}$.

Morphisms of polynomial functors

Let P and P' be two polynomial functors from I to J. A morphism F of polynomial functors is given by the following data:

- a function $f : B \to B'$ such that $\mathtt{t}' \circ f = \mathtt{t}$,
- and for each b a *bijection*

$$f_b : \mathtt{A}^P(b) \to \mathtt{A}^{P'}(f(b))$$

  such that $\mathtt{s}'_{f(b)} \circ f_b = \mathtt{s}_b$.

The (vertical) composition of morphisms is defined in the obvious way:

$$(g, (g_{b'})_{b' \in B'}) \circ (f, (f_b)_{b \in B}) = (g \circ f, (g_{f(b)} \circ f_b)_{b \in B})$$

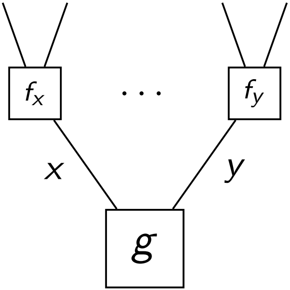
Composition of polynomial functors

Let $P : I \to J$, $Q : J \to K$. We set $Q \circ P = R$, where B^R is the set of all formal "explicit simultaneous substitutions" that are well-typed according to the following rule:

$$\frac{(X, s) \vdash^Q g : k \qquad (\forall x \in X,\ (Y_x, s_x) \vdash^P f_x : s(x))}{(\sum_{x \in X} Y_x, [s_x]_{x \in X}) \vdash^R g\{x \leftarrow f_x \mid x \in X\} : k}$$

where $[s_x]_{x \in X}$ is the copairing of all s_x's.

Composition of polynomial functors (pictorially)



Identity polynomial functor

$Id_I : I \to I$ is defined by setting, for every element $i \in I$:

$$\frac{}{(\{*\}, * \mapsto i) \vdash i : i}$$

Polynomial functors are the 1-morphisms of a

bicategory

(composition is associative only up to isomorphism).

Polynomial monad (simultaneous multiplication)

A polynomial monad is given by a polynomial endofunctor $P : I \to I$, together with a mutliplication from $P \circ P$ to P, and a unit from Id_I to P. Notation for (simultaneous) multiplication:

$$f \circ \{g_x \mid x \in \mathtt{A}(f)\} \quad \text{or} \quad f \circ \{x \leftarrow g_x \mid x \in \mathtt{A}(f)\}$$

$$\chi_{f\{x \leftarrow g_x \mid x \in \mathtt{A}(f)\}} : \sum_{x \in \mathtt{A}(f)} \mathtt{A}(g_x) \to \mathtt{A}(f \circ \{g_x \mid x \in \mathtt{A}(f)\})$$

(the bijection $\chi_{f\{x \leftarrow g_x \mid x \in X\}}$ indicates how the inputs of $f \circ \{g_x \mid x \in \mathtt{A}(f)\}$ match those of the the g_x's).

Notation for the unit: $\eta(i) = \mathtt{id}_i \quad (A_i, s_i) \vdash \mathtt{id}_i : i$, where A_i is a singleton whose unique element is mapped by s_i to i.

The multiplication and unit have to satisfy the equalities of a monoid, and the associated bijections must satisfy some coherence conditions.

Polynomial monads versus operads

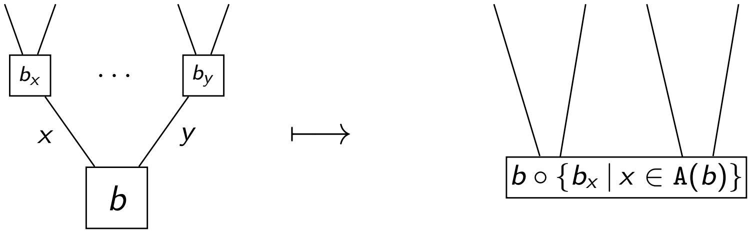
Polynomial monads are a version of (set) operads that are

- non-symmetric
- non-skeletal (inputs are named, rather than numbered)
- coloured

Note that the mechanics of polynomial functors dictates that the renaming of wires after composition be specified as part of the data defining the structure.

Polynomial monads are exactly the version of multicategories given by. Hermida, Makkai and Power.

Simultaneous multiplication pictorially

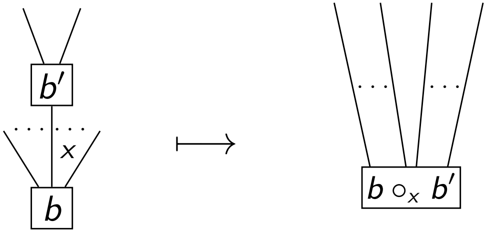


Polynomial monad (individual multiplication)

Equivalently, we can specify a multiplication of two operations along one chosen input of the first one.
Formally, one specifies an operation mapping each triple (f, x, g) such that $x \in \mathtt{A}(f)$ and $\mathtt{t}(g) = \mathtt{s}_f(x)$ to a triple $(f \circ_x g, \phi, \psi)$ such that

$(Z, \mathtt{s}_Z) \vdash f \circ_x g : \mathtt{t}(f)$ $\quad$ $\phi : \mathtt{A}(f) \setminus \{x\} \to Z \quad \psi : \mathtt{A}(g) \to Z$,
ϕ and ψ commute with source functions, and
$[\phi, \psi] : \mathtt{A}(f) \setminus \{x\} + \mathtt{A}(g) \to Z$ is a bijection

Individual multiplication pictorially



Between individual and simultaneous

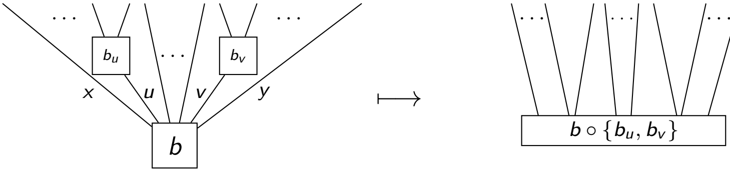
We shall be working with an equivalent *intermediate notion* of multiplication, neither simultaneous nor individual, parameterised by any subset Y of $\mathtt{A}(f)$:

$$f \circ \{g_y \mid y \in Y\}$$

$$\chi_{f\{y \leftarrow g_y \mid y \in Y\}} : (\mathtt{A}(f) \setminus Y) + \sum_{y \in Y} \mathtt{A}(g_y) \ \rightarrow \ \mathtt{A}(f \circ \{g_y \mid y \in Y\})$$

In the degenerate case where Y is empty, we set $f \circ \{\} = f$ and $\chi_{f\{Y\}} = id$.

Intermediate multiplication pictorially



Free polynomial monad (trees)

Let $P : I \to I$. We define a new polynomial endofunctor $P^\star$ on I. The operations of $P : I \to I$ are trees

- whose nodes are decorated by operations of P, and
- whose edges are decorated in I .

Syntactically, they are those terms from the raw syntax

$$\begin{aligned} T &::= \langle \underline{i} \rangle \parallel \mathbb{T} \\ \mathbb{T} &::= \underline{b}\{y \leftarrow \mathbb{T}_y \mid x \in Y\} \qquad \text{(where } Y \text{ is a subset of } \mathtt{A}(b)\text{)} \end{aligned}$$

that are accepted by the following typing system:

$$\frac{}{(\{*\}, * \to i) \vdash^{P^\star} \langle \underline{i} \rangle : i}$$

$$\frac{(X, s) \vdash^{P} b : i \qquad Y \subseteq X \qquad (\forall y \in Y \;\; (L_y, s_y) \vdash^{P^\star} \mathbb{T}_y : s(y))}{((X \setminus Y) + \sum_{y \in Y} L_y \,,\, [s_{\restriction X \setminus Y}, [s_y]_{y \in Y}]) \vdash^{P^\star} \underline{b}\{y \leftarrow \mathbb{T}_y \mid y \in Y\} : i}$$

where the symbol $\restriction$ denotes restriction.

Free polynomial monad (arities)

The typing system builds the arities of our trees by induction.

Fact: In $P^{\star}$, the arity of a tree T is the set of the occurrences of its **leaves**, which are each unambiguously identified by following the sequence of the names of the edges leading to it.
We call such sequences addresses.

Monad structure of $P^\star$

- We set $\mathtt{id}_i = \langle \underline{i} \rangle$
- We define $S \circ_l T$ recursively, as follows:

$$\frac{(L, s) \vdash T : i}{\langle \underline{i} \rangle \circ_* T = T}$$

$$\frac{z \in Y \qquad \mathbb{S}'_z = \mathbb{S}_z \circ_l T \qquad \forall y \neq z \ \ \mathbb{S}'_y = \mathbb{S}_y}{\underline{b}\{y \leftarrow \mathbb{S}_y \mid y \in Y\} \circ_{(z,l)} T = \underline{b}\{y \leftarrow \mathbb{S}'_y \mid y \in Y\}}$$

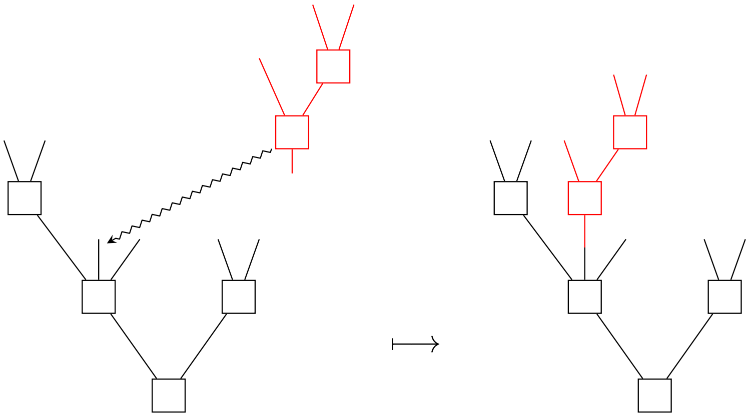
$$\frac{z \notin Y \qquad \mathbb{S}'_z = T}{\underline{b}\{y \leftarrow \mathbb{S}_y \mid y \in Y\} \circ_{(y,l)} T = \underline{b}\{y \leftarrow \mathbb{S}'_y \mid y \in Y \cup \{z\}\}}$$

$S \circ_l T$ is the tree obtained by

grafting T on the leaf of address l of S.

Note that in the process, the leaves of T are all renamed in $S \circ_l T$.

The star multiplication (pictorially)



A consequence of the freeness of $P^\star$

$P \mapsto P^\star$ is left adjoint to the forgetful functor from polynomial monads to polynomial endofunctors (over a fixed set I of colours).

If P is already a polynomial monad, then the freeness of $P^\star$ induces

- an intepretation morphism $(T \mapsto [\![T]\!]^{P^\star}) : P^\star \to P$:

$$[\![\langle \underline{i} \rangle]\!]^{P^\star} = \mathtt{id}_i$$
$$[\![\underline{b}\{y \leftarrow \mathbb{T}_y \mid y \in Y\}]\!]^{P^\star} = b \circ^P \{[\![\mathbb{T}_y]\!]^{P^\star} \mid y \in Y\}$$

- with associated bijections ρ_T, each from the set of leaves of T to the arity of $[\![T]\!]^{P^\star}$:

$$\rho_{\underline{b}\{y \leftarrow \mathbb{T}_y \mid y \in Y\}}(yl) = \chi^P_{b\{y \leftarrow [\![\mathbb{T}_y]\!] \mid y \in Y\}}(y, \rho_{\mathbb{T}_y}(l))$$
$$\rho_{\underline{b}\{y \leftarrow \mathbb{T}_y \mid y \in Y\}}(z) = \chi^P_{b\{y \leftarrow [\![\mathbb{T}_y]\!] \mid y \in Y\}}(z)$$

Another monad on trees: the + construction (Baez-Dolan)

Here we follow (Kock, Joyal, Batanin,and Mascari 2010).

We again suppose that P is a polynomial monad on some I. Then $\mathtt{Tr}(P) = B^{P^\star}$ gives rise to

another polynomial monad P^+, not on I, but on $B = B^P$:

- $B^{P^+} = B^{P^\star}$
- The arity of a tree is not its set of leaves anymore, but its set of **nodes**. Formally:

$$\mathtt{A}^{P^+}(\langle \underline{i} \rangle) = \emptyset$$
$$\mathtt{A}^{P^+}(\underline{b}\{y \leftarrow \mathbb{T}_y \mid y \in Y\}) = \{\epsilon\} + \sum_{y \in Y} \mathtt{A}^{P^+}(\mathbb{T}_y)$$

$$s^{P^+}_{\underline{b}\{y \leftarrow \mathbb{T}_y \mid y \in Y\}}(\epsilon) = b$$
$$s^{P^+}_{\underline{b}\{y \leftarrow \mathbb{T}_y \mid y \in Y\}}(z\, n) = s^{P^+}_{\mathbb{T}_z}(n) \qquad (z \in Y)$$

- The target colour of T is $[\![T]\!]^{P^\star}$

Contrasting the $\star$ and the $+$ constructions

- For a polynomial functor on I, P^* is a polynomial monad on I,
- For a polynomial monad on I, P^* is a polynomial monad on B^P.

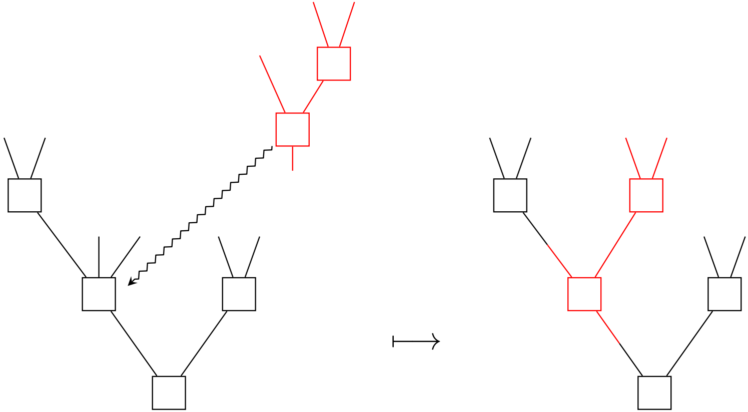
We have

$$B^{P^\star} = \mathtt{Tr}(P) = B^{P^+}$$

but arities are different:

- leaves (colored in I)
- versus nodes (colored in B^P)

The + multiplication (pictorially)



The + composition

$$\frac{\llbracket T \rrbracket = b}{\underline{b}\{x \leftarrow \mathbb{S}_y \mid y \in Y\} \circ_{\epsilon}^{P^+} T = T \circ^{P^\star} \{l \leftarrow \mathbb{S}_{\rho_T^{P^\star}(l)} \mid \rho_T^{P^\star}(l) \in Y\}}$$

$$\frac{z \in Y \qquad \mathbb{S}'_z = \mathbb{S}_z \circ_n^{P^+} T \qquad (\forall y \neq z \;\; \mathbb{S}'_y = \mathbb{S}_y)}{\underline{b}\{y \leftarrow \mathbb{S}_y \mid y \in Y\} \circ_{z\,n}^{P^+} T = \underline{b}\{y \leftarrow \mathbb{S}'_y \mid y \in Y\}}$$

$$\chi^{P^+}_{\mathbb{S}\{n_0 \leftarrow T\}}(n.1) = n \qquad (n_0 \text{ is not a prefix of } n)$$

$$\chi^{P^+}_{\mathbb{S}\{n_0 \leftarrow T\}}((n_0 x n_1).1) = n_0\,((\rho_T^{P^\star})^{-1}(x))\,n_1$$

$$\chi^{P^+}_{\mathbb{S}\{n_0 \leftarrow T\}}((n_0 x n_1).1) = n_0\,n$$

An iterative architecture (with its bookkeeping)

From a polynomial monad P (with $\mathtt{id}^P$, $\circ^P$, χ^P):

- Build the $P^\star$ structure:
  - $\circ^{P^\star}$ (using just the polynomial functor structure of P)
  - $[\![T]\!]^{P^\star}$ needs $\mathtt{id}^P$ and $\circ^P$
  - $\rho^{P^\star}$ needs χ^P
- Build the P^+ structure:
  - $\circ$ needs $\circ^{P^\star}$ and $\rho^{P^\star}$
  - χ^{P^+} needs $\rho^{P^\star}$

Opetopes (definition)

Opetopes are defined by iteration of the $+$ construction.

- Basis $=$ identity polynomial functor on a singleton set.

$$B^0 = \{\blacklozenge\} \qquad B^1 = \{\blacksquare\}$$

B^0 is just a set, and B^1 features a polynomial monad $\mathcal{O}^1$ on B^0:

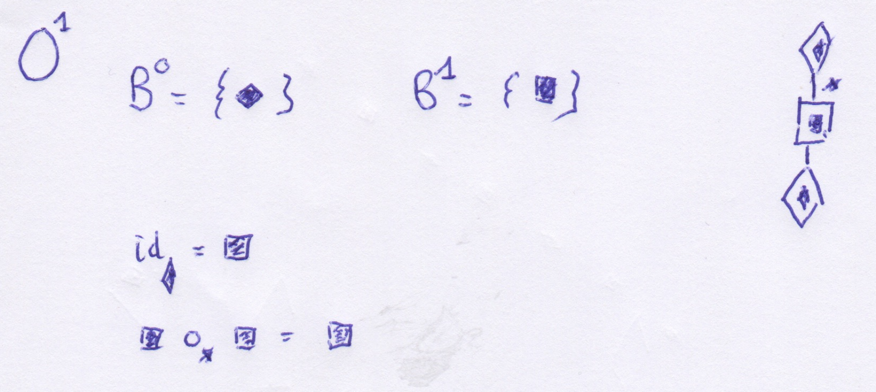
$$(\{*\}, * \mapsto \blacklozenge) \vdash \blacksquare : \blacklozenge$$

The monad structure is given by setting $\blacksquare \circ_* \blacksquare = \blacksquare$.

- Induction: We set

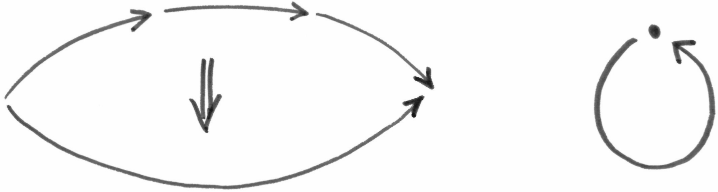
$$\mathcal{O}^n = (\mathcal{O}^{n-1})^+$$

The unique 1-opetope



2-opetopes (sequences of arrows)

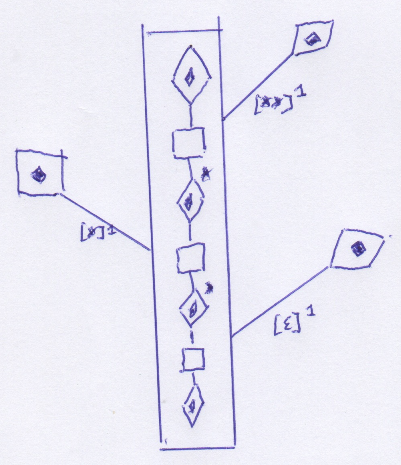
2-opetopes describe shapes of generating 2-cells, from a sequence of arrows to an arrow:

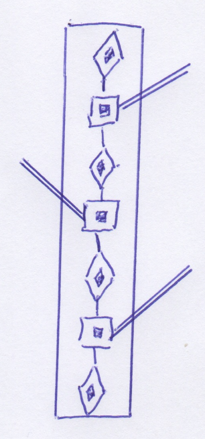


$$\begin{cases} [\epsilon]^1 \cdot \blacksquare \\ [*]^1 \cdot \blacksquare \\ [**]^1 \cdot \blacksquare \end{cases}$$

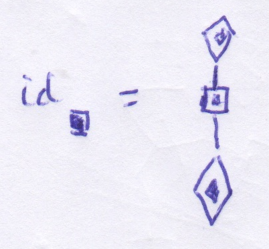
$$\langle \blacklozenge \rangle$$

Syntax versus Pictogrammes



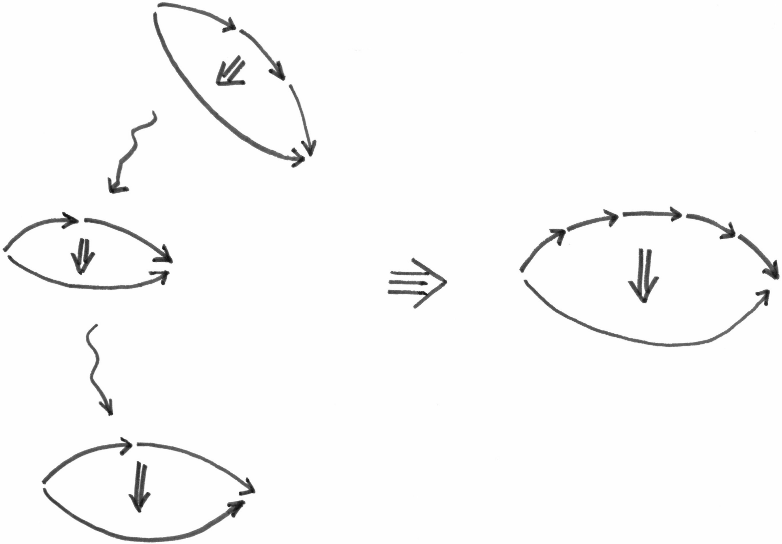


The monad structure on 2-opetopes



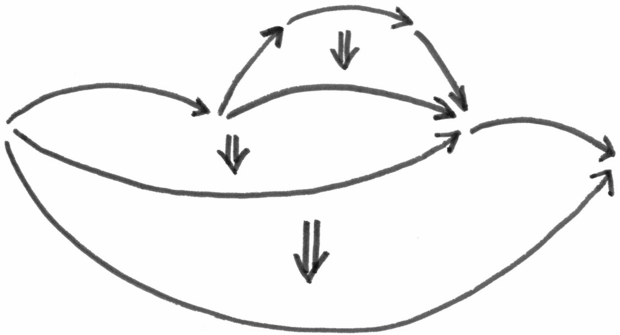
$$\left\{ \begin{array}{l} [\epsilon]^1 \cdot \blacksquare \\ [*]^1 \cdot \blacksquare \\ [**]^1 \cdot \blacksquare \end{array} \right. \circ_{[*]^1} \left\{ \begin{array}{l} [\epsilon]^1 \cdot \blacksquare \\ [*]^1 \cdot \blacksquare \end{array} \right. = \left\{ \begin{array}{l} [\epsilon]^1 \cdot \blacksquare \\ [*]^1 \cdot \blacksquare \\ [**]^1 \cdot \blacksquare \\ [***]^1 \cdot \blacksquare \end{array} \right.$$

The shape of a 3-opetope

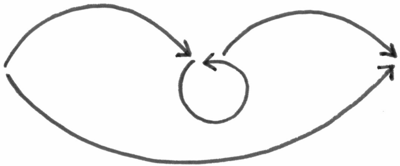


3-opetopes (shortened as pasting diagrams)

The information of a 3-opetope can be captured by a pasting diagram:



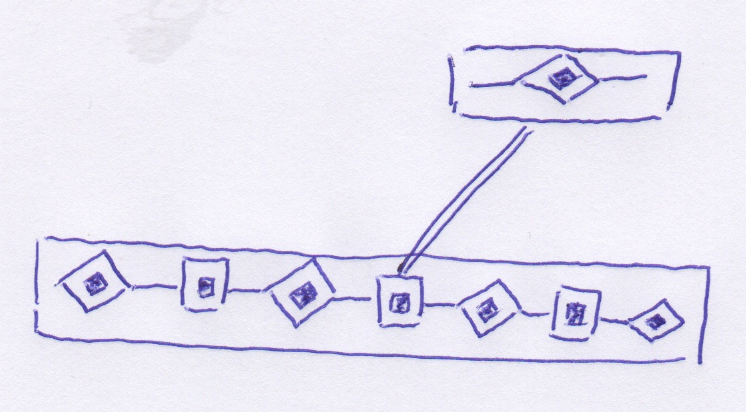
3-opetopes (example 1)



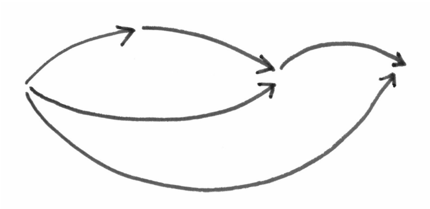
$$\begin{cases} [\epsilon]^2 \cdot \begin{cases} [\epsilon]^1 \cdot \blacksquare \\ [*]^1 \cdot \blacksquare \\ [**]^1 \cdot \blacksquare \end{cases} \\ [[*]^1]^2 \cdot \langle \blacklozenge \rangle \end{cases}$$

Think of $\blacksquare$ as an arrow, and of $\blacklozenge$ as a point (and then of $\langle \blacklozenge \rangle$ as a degeneracy).

Example 1 in pictogramme form



3-opetopes (example 2)

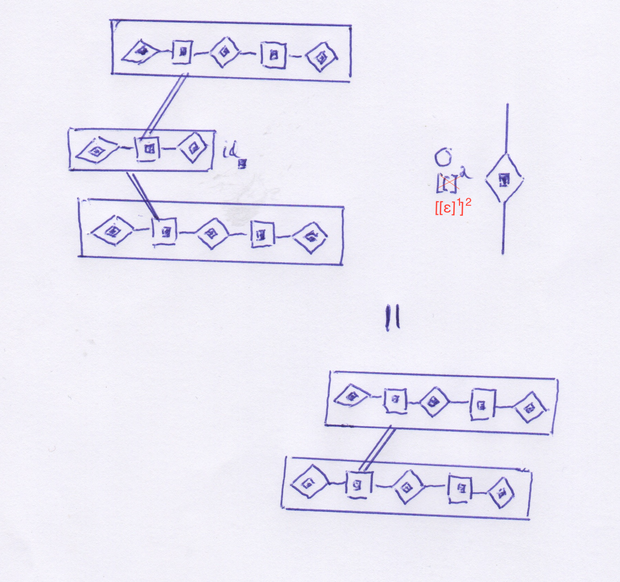


$$\left\{ \begin{array}{l} [\epsilon]^2 \cdot \left\{ \begin{array}{l} [\epsilon]^1 \cdot \blacksquare \\ [*]^1 \cdot \blacksquare \end{array} \right. \\ [[\epsilon]^1]^2 \cdot \ \left\{ \ [\epsilon]^1 \cdot \blacksquare \ [*]^1 \cdot \blacksquare \right. \end{array} \right.$$

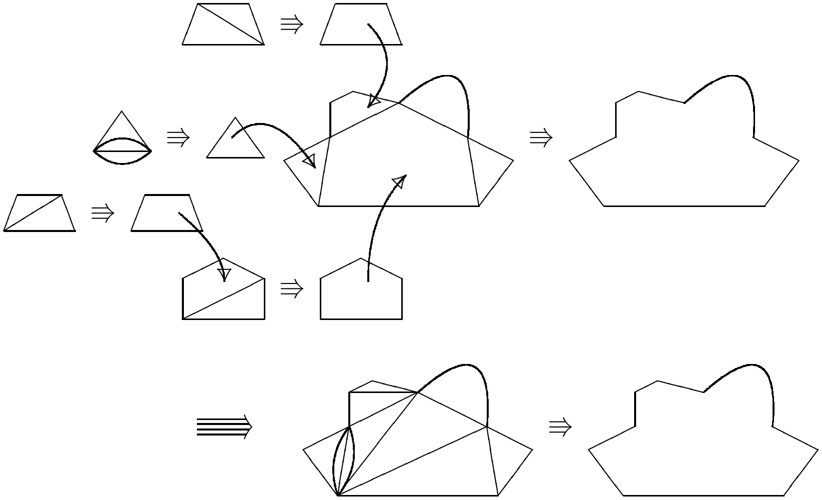
3-opetopes (example 3)

$$
\left\{
\begin{array}{l}
[\epsilon]^2 \cdot \left\{ \begin{array}{l} [\epsilon]^1 \cdot \blacksquare \\ [*]^1 \cdot \blacksquare \\ [**]^1 \cdot \blacksquare \end{array} \right. \\
[[**]^1]^2 \cdot \left\{ \; [\epsilon]^1 \cdot \blacksquare \right. \\
[[*]^1]^2 \cdot \langle \blacklozenge \rangle \\
[[\epsilon]^1]^2 \cdot \left\{ \begin{array}{l} [\epsilon]^1 \cdot \blacksquare \\ [*]^1 \cdot \blacksquare \end{array} \right. \\
[[\epsilon]^1[\epsilon^1]]^2 \cdot \left\{ \; [\epsilon]^1 \cdot \blacksquare \right.
\end{array}
\right.
$$

A composition of 3-opetopes



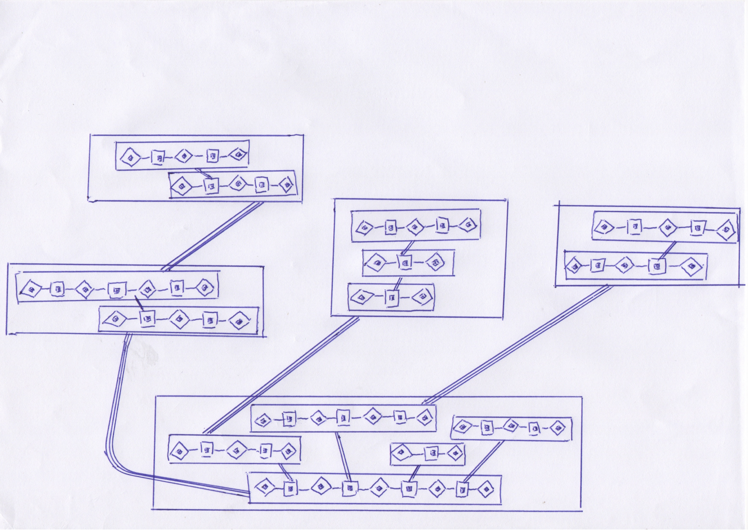
An example of a 4-opetope (drawing)



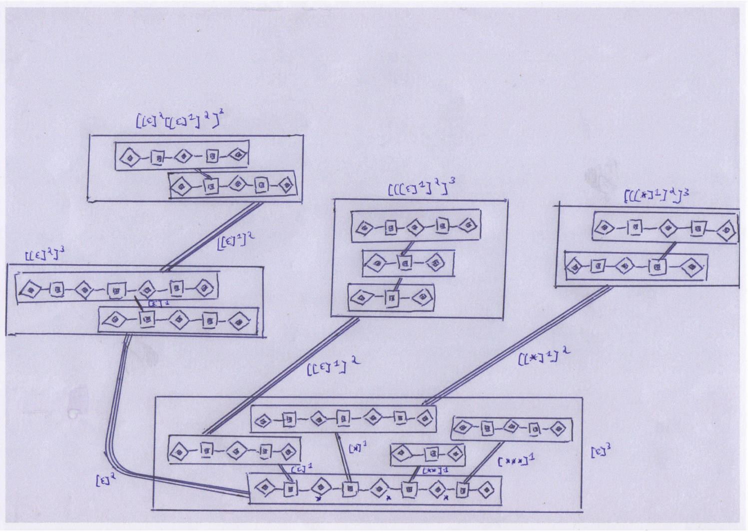
An example of a 4-opetope (encoding)

$$
\left\{
\begin{array}{l}
[\epsilon]^3 \cdot \left\{
\begin{array}{l}
[\epsilon]^2 \cdot \{[\epsilon]^1 \cdot \blacksquare, [*]^1 \cdot \blacksquare, [**]^1 \cdot \blacksquare, [***]^1 \cdot \blacksquare\} \\
[[\epsilon]^1]^2 \cdot \{[\epsilon]^1 \cdot \blacksquare, [*]^1 \cdot \blacksquare\} \\
[[*]^1]^2 \cdot \{[\epsilon]^1 \cdot \blacksquare, [*]^1 \cdot \blacksquare, [**]^1 \cdot \blacksquare\} \\
[[**]^1]^2 \cdot \{[\epsilon]^1 \cdot \blacksquare\} \\
[[***]^1]^2 \cdot \{[\epsilon]^1 \cdot \blacksquare, [*]^1 \cdot \blacksquare\}
\end{array}
\right. \\
[[\epsilon]^2]^3 \cdot \left\{
\begin{array}{l}
[\epsilon]^2 \cdot \{[\epsilon]^1 \cdot \blacksquare, [*]^1 \cdot \blacksquare\} \\
[[\epsilon]^1]^2 \cdot \{[\epsilon]^1 \cdot \blacksquare, [*]^1 \cdot \blacksquare, [**]^1 \cdot \blacksquare\}
\end{array}
\right. \\
[[[\epsilon]^1]^2]^3 \cdot \left\{
\begin{array}{l}
[\epsilon]^2 \cdot \{[\epsilon]^1 \cdot \blacksquare\} \\
[[\epsilon]^1]^2 \cdot \{[\epsilon]^1 \cdot \blacksquare\} \\
[[\epsilon]^1[\epsilon]^1]^2 \cdot \{[\epsilon]^1 \cdot \blacksquare, [*]^1 \cdot \blacksquare\}
\end{array}
\right. \\
[[[*]^1]^2]^3 \cdot \left\{
\begin{array}{l}
[\epsilon]^2 \cdot \{[\epsilon]^1 \cdot \blacksquare, [*]^1 \cdot \blacksquare\} \\
[[*]^1]^2 \cdot \{[\epsilon]^1 \cdot \blacksquare, [*]^1 \cdot \blacksquare\}
\end{array}
\right. \\
[[\epsilon]^2[[\epsilon]^1]^2]^3 \cdot \left\{
\begin{array}{l}
[\epsilon]^2 \cdot \{[\epsilon]^1 \cdot \blacksquare, [*]^1 \cdot \blacksquare\} \\
[\epsilon]^1]^2 \cdot \{[\epsilon]^1 \cdot \blacksquare, [*]^1 \cdot \blacksquare\}
\end{array}
\right.
\end{array}
\right.
$$

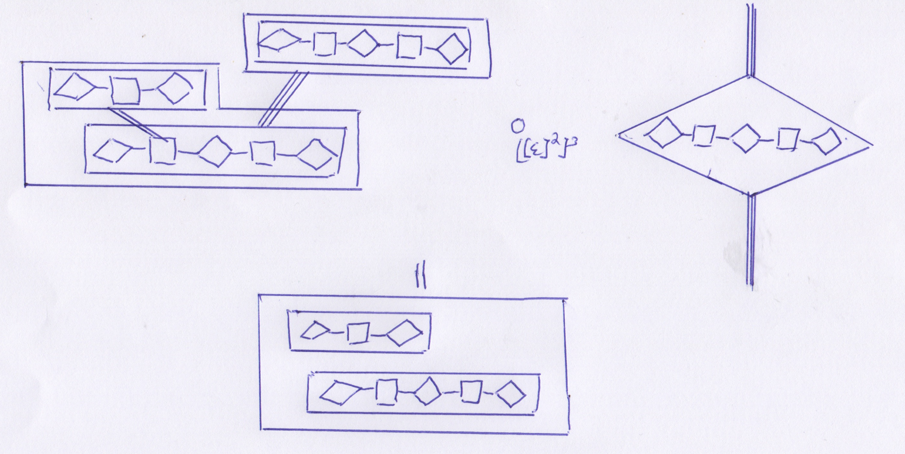
An example of a 4-opetope (pictogram)



An example of a 4-opetope (from pictogram to syntax)



A composition of 4-opetopes



What next

This is work in progress. We want to test the usefulness and readability of this notation (opetopes are notoriously difficult to work with).

Experience will tell!

This work seems closest to the work of Hermida, Makkai and Power, who lacked the language of polynomial monads.