Propagation of chaos for aggregation equations with no-flux boundary conditions and sharp sensing zones

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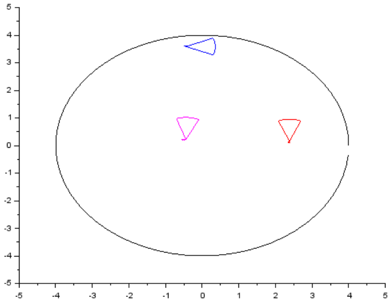
PDE/probability interactions : kinetic equations, large time and propagation of chaos CIRM, april 18-22, 2017



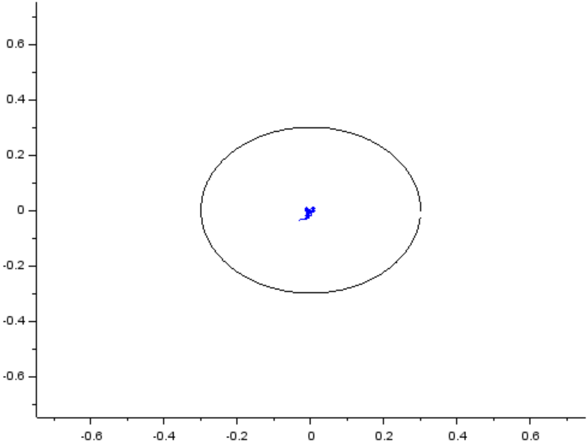
- Each agent an is submitted independent diffusion and remain in a convex bounded domain $\mathcal{O}$.
- An agent j at position X_j lies in the vision cone $C(X_i)$ of another agent i at position X_i, then the action of j on i is given by $\nabla\phi(X_i - X_j)$
- I.I.D. initial conditions.

Skorokhod's problem I

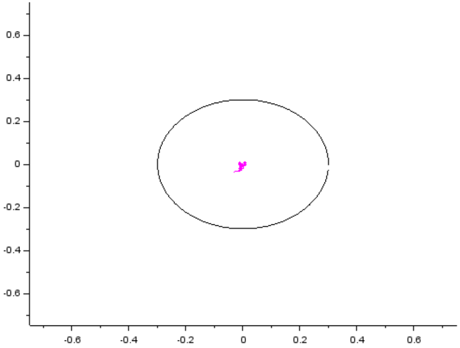
Consider $\mathcal{O}$ an open convex subset of $\mathbb{R}^d$ (with $0 \in \mathcal{O}$), and $(B_t)_{t\geq 0}$ be d-dimensional Brownian motion. There exists a unique pair of processes $(X_t, K_t)_{t\geq 0}$ such that

- $\forall t \geq 0,\ X_t = B_t - K_t$
- The process $(X_t)_{t\geq 0}$ remains in $\mathcal{O}$.
- $$K_t = \int_0^t n(X_s) d|K|_s \quad , \quad |K|_t = \int_0^t \mathbf{1}_{\partial\mathcal{O}}(X_s) d|K|_s,$$
  where n is the *outward normal* of $\partial\mathcal{O}$, and $|\cdot|_t$ is the *total variation* of the process at time t.

Skorokhod's problem II



Skorokhod's problem III



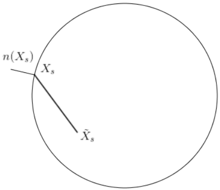
$$\begin{cases} X_t = B_t - K_t \in \mathcal{O}, \quad K_t = \int_0^t n(X_s)\, d|K|_s, \quad |K|_t = \int_0^t \mathbf{1}_{\partial\mathcal{O}}(X_s)\, d|K|_s, \\ x^\epsilon_{t_{n+1}} = x^\epsilon_{t_n} + \sqrt{\Delta t}\, U_n + \frac{\Delta t}{\epsilon} \nabla p(x^\epsilon_{t_n}), \quad \mathcal{L}(U_n) = \mathcal{N}(0,1), \quad p(x) = d^2(x, \mathcal{O}) \end{cases}$$

- Existence and uniqueness of solution to the SDE

$$\begin{cases} X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s) \cdot \, dB_s - K_t, \\ K_t = \int_0^t n(X_s)\, d|K|_s, \quad |K|_t = \int_0^t \mathbf{1}_{\partial\mathcal{O}}(X_s)\, d|K|_s, \end{cases}$$

- (Tanaka '79) use the convexity of $\mathcal{O}$

$$|X_t - \tilde{X}_t|^2 = \cdots - \int_0^t \underbrace{\left\langle X_s - \tilde{X}_s, n(X_s) \right\rangle d|K|_s}_{\geq 0} - 2\int_0^t \underbrace{\left\langle \tilde{X}_s - X_s, n(\tilde{X}_s) \right\rangle d|\tilde{K}|_s}_{\geq 0},$$



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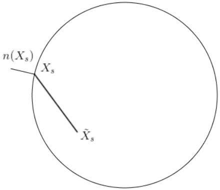

N-Particles system and nonlinear limit I

N-Particles system

$$\begin{cases} X_t^i = X_0^i + \int_0^t \frac{1}{N} \sum_{j=1}^N \mathbf{1}_{C(X_s^i)}(X_s^j - X_s^i) \nabla\phi(X_s^i - X_s^j)\, ds + \sqrt{2\sigma} B_t^i - K_t^i, \\ K_t^i = \int_0^t n(X_s^i)\, d|K^i|_s, \quad |K^i|_t = \int_0^t \mathbf{1}_{\partial\mathcal{O}}(X_s^i)\, d|K^i|_s, \end{cases} \tag{1}$$

where $C(\cdot)$ is a vision cone of angle $\alpha \in (0, 2\pi)$ and radius $r > 0$ given by

$$C(x) = \{z \in \mathbb{R}^d \;\; |z| \leq r \;, \frac{\langle z, w(x)\rangle}{|z||w(x)|} \geq \cos(\frac{\alpha}{2})\},$$

w is a Lipschitz orientational field, and $\nabla\phi$ is a bounded Lipschitz interaction field.

N-Particles system and nonlinear limit II

Nonlinear limit

$$\begin{cases} Y_t = Y_0 + \int_0^t \int_{\mathcal{O}} \mathbf{1}_{C(Y_s)}(y - Y_s)\nabla\phi(Y_s - y)\rho_s(dy)\, ds + \sqrt{2\sigma} B_t - K_t, \\ \rho_s = \mathcal{L}(Y_s), \\ K_t = \int_0^t n(Y_s)\, d|K|_s, \quad |K|_t = \int_0^t \mathbf{1}_{\partial\mathcal{O}}(Y_s)\, d|K|_s, \end{cases}$$

The family $(\rho_t)_{t\geq 0}$ solves

$$\begin{cases} \partial_t \rho_t + \nabla \cdot (\rho_t V[\rho_t]) = \sigma \Delta \rho_t, \quad x \in \mathcal{O}, \quad t > 0, \\ V[\rho](x) = \int_{\mathcal{O}} \mathbf{1}_{C(x)}(y - x)\nabla\phi(x - y)\rho(dy), \\ \langle \sigma\nabla\rho - \rho V[\rho], n \rangle = 0 \quad \text{on} \quad \partial\mathcal{O}. \end{cases} \tag{2}$$

Theorem (Choi, S., 2016)

Assume that $(X_0^i)_{i=1,\cdots,N}$ *the initial conditions to* (1) *are i.i.d. of law* $\rho_0 \in (\mathcal{P}_1 \cap L^\infty)(\mathcal{O})$ *and let* ρ *be the solution to equation* (2) *with initial data* ρ_0*. Then, for any* $\epsilon \in (0, 1/2)$ *and* $p > d/2$*, there exists a constant* $C_{\epsilon,p,d} > 0$ *such that*

$$\sup_{s\in[0,t]} \mathbb{E}[\mathcal{W}_p(\mu_s^N, \rho_s)] \le C_{\epsilon,p,d}\, e^{C_{\epsilon,p,d}\int_0^t \|\rho_s\|_{L^1\cup L^\infty}\, ds}\left(N^{-\frac{1}{2}+\epsilon} + N^{-\frac{1}{2p}}\right),$$

where $\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$ *is the empirical measure associated to the particle system* (1)*, and* $\mathcal{W}_p$ *is the Wasserstein metric of order* p *and* $\|\cdot\|_{L^1\cup L^\infty} = \|\cdot\|_{L^1} + \|\cdot\|_{L^\infty}$.

Sketch of proof

Nonlinear particles system

$$\begin{cases} Y_t^i = X_0^i + \int_0^t \int_{\mathcal{O}} \mathbf{1}_{C(Y_s^i)}(y - Y_s^i)\nabla\phi(Y_s^i - y)\rho_s(dy)\, ds + \sqrt{2\sigma} B_t^i - \tilde{K}_t^i, \\ \rho_s = \mathcal{L}(Y_s^i), \\ \tilde{K}_t^i = \int_0^t n(Y_s^i)\, d|\tilde{K}^i|_s, \quad |\tilde{K}^i|_t = \int_0^t \mathbf{1}_{\partial\mathcal{O}}(Y_s^i)\, d|\tilde{K}^i|_s, \end{cases}$$

and the associated empirical measure $\nu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{Y_t^i}$.

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- $$\mathcal{W}_p(\mu_t^N, \rho_t) \leq \mathcal{W}_\infty(\mu_t^N, \nu_t^N) + \mathcal{W}_p(\nu_t^N, \rho_t)$$
- Since the $(Y_t^i)_{i=1,\cdots,N}$ are i.i.d., we use a Theorem by Fournier and Guillin to get

$$\mathbb{E}[\mathcal{W}_p(\nu_t^N, \rho_t)^p] \leq C_{p,d} N^{-\frac{1}{2}},$$

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Sketch of proof I

- Aim to get a Gronwall's inequality for

$$\mathcal{W}_\infty(\mu_t^N, \nu_t^N) = \min_{\sigma \in \mathcal{S}^N} \max_{i=1,\cdots,N} |X_t^i - Y_t^{\sigma(i)}| \leq \max_{i=1,\cdots,N} |X_t^i - Y_t^i| := A_t,$$

-

$$X_t^i - Y_t^i = \int_0^t \frac{1}{N}\sum_{j=1}^N \mathbf{1}_{C(X_s^i)}(X_s^j - X_s^i) - \int_{\mathcal{O}} \mathbf{1}_{C(Y_s^i)}(y - Y_s^i)\rho_s(dy)\, ds$$
$$- (K_t^i - \tilde{K}_t^i)$$

- We get arid of the term $(K_t^i - \tilde{K}_t^i)$, thanks to the geometry of the domain $\mathcal{O}$ (convexity or exterior-sphere condition).

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$$\Big|\frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{C(X_s^i)}(X_s^j - X_s^i) - \int_{\mathcal{O}}\mathbf{1}_{C(Y_s^i)}(y - Y_s^i)\rho_s(dy)\Big| =$$

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$$+\overbrace{\Big|\frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{C(Y_s^i)}(Y_s^j - Y_s^i) - \int_{\mathcal{O}}\mathbf{1}_{C(Y_s^i)}(y - Y_s^i)\rho_s(dy)\Big|}^{:=H_s^{i,N}}$$

$$= \frac{1}{N}\sum_{j=1}^{N}\Big|\mathbf{1}_{C(X_s^i)}(X_s^j - X_s^i) - \mathbf{1}_{C(X_s^i)}(Y_s^j - Y_s^i)\Big|$$

$$+ \frac{1}{N}\sum_{j=1}^{N}\Big|\mathbf{1}_{C(X_s^i)} - \mathbf{1}_{C(Y_s^i)}\Big|(Y_s^j - Y_s^i) + H_s^{i,N}$$

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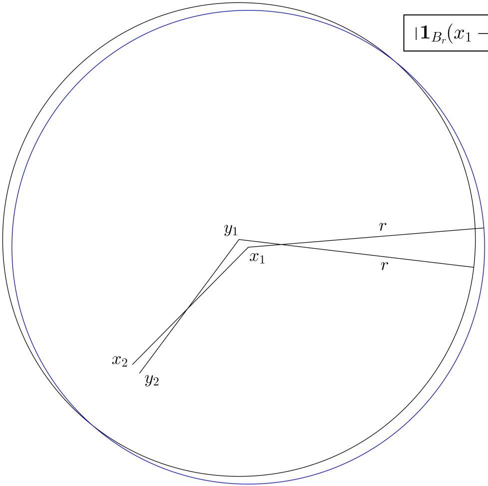
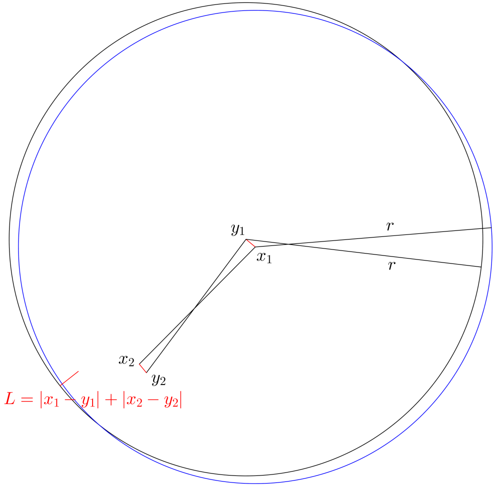
Rope argument



$$|\mathbf{1}_{B_r}(x_1 - x_2) - \mathbf{1}_{B_r}(y_1 - y_2)| \leq ??$$

Rope argument



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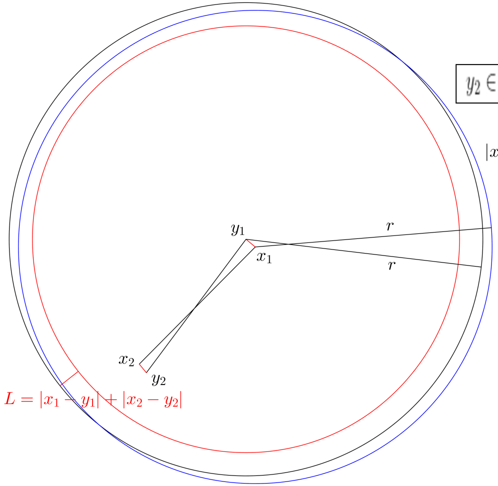


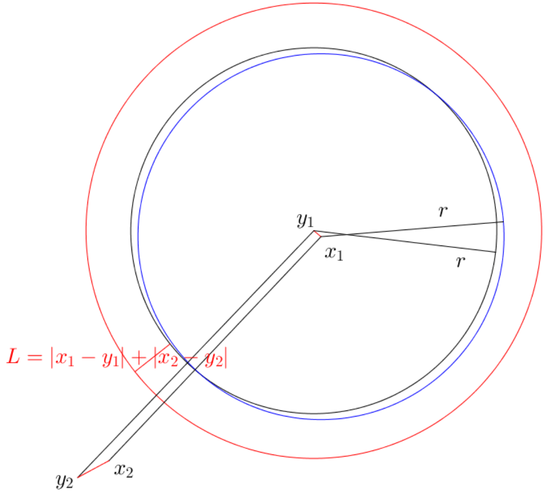
$$\boxed{y_2 \in B(r - L, y_1) \Rightarrow x_2 \in B(r, x_1)}$$

$$|x_1 - x_2| \le |y_1 - y_2|$$

$$+ \overbrace{|x_1 - y_1| + |x_2 - y_2|}^{=L}$$

$$\le r$$

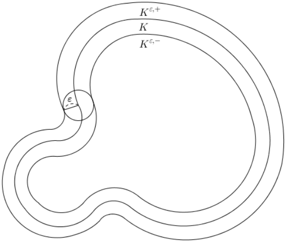
Rope argument



$$y_2 \notin B(r+L, y_1) \Rightarrow x_2 \notin B(r, x_1)$$

- $$y_1 - y_2 \notin B_{r+L} \setminus B_{r-L} := C_r^L \Rightarrow \mathbf{1}_{B_r}(y_1 - y_2) - \mathbf{1}_{B_r}(x_1 - x_2) = 0,$$
  $$L = |x_1 - y_1| + |x_2 - y_2|.$$

- $$|\mathbf{1}_{B_r}(x_1 - x_2) - \mathbf{1}_{B_r}(y_1 - y_2)| \leq \mathbf{1}_{C_r^{|x_1 - y_1| + |x_2 - y_2|}}(y_1 - y_2).$$
  $$|\mathbf{1}_K(x_1 - x_2) - \mathbf{1}_K(y_1 - y_2)| \leq \mathbf{1}_{\partial^{|x_1 - y_1| + |x_2 - y_2|}K}(y_1 - y_2).$$



$$\partial^\varepsilon K = K^{\varepsilon,+} \setminus K^{\varepsilon,-} \quad , Vol(\partial^\varepsilon K) \simeq 2\varepsilon|\partial K|.$$

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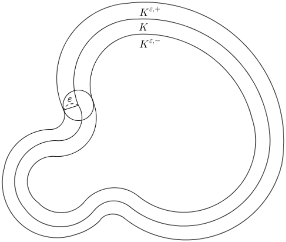

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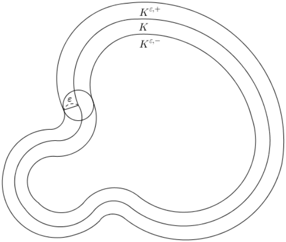

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$$
\begin{aligned}
&\frac{1}{N}\sum_{j=1}^{N}\left|\mathbf{1}_{C(X_s^i)}(X_s^j - X_s^i) - \mathbf{1}_{C(X_s^i)}(Y_s^j - Y_s^i)\right| \\
&\qquad \leq \frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{\partial^{|X_s^j - Y_s^j| + |X_s^i - Y_s^i|}C(X_s^i)}(Y_s^j - Y_s^i) \\
&\qquad \leq \frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{\partial^{2\max_{k=1,\cdots,N}|X_s^k - Y_s^k|}C(X_s^i)}(Y_s^j - Y_s^i) \\
&\qquad = \int_{\mathcal{O}} \mathbf{1}_{\partial^{2A_s}C(X_s^i)}(y - Y_s^i)\,\nu_s^N(dy) \\
&\qquad = \int_{\mathcal{O}} \mathbf{1}_{\partial^{2A_s}C(X_s^i)}(y - Y_s^i)\,\rho_s(dy) + \overbrace{\left|\int_{\mathcal{O}} \mathbf{1}_{\partial^{2A_s}C(X_s^i)}(y - Y_s^i)\,(\rho_s - \nu_s^N)(dy)\right|}^{=J_s^{i,N}} \\
&\qquad \leq \|\rho_s\|_{L^\infty} Vol(\partial^{2A_s}C(X_s^i)) + J_s^{i,N} \leq C\|\rho_s\|_{L^\infty}A_s + J_s^{i,N}
\end{aligned}
$$

$$\frac{1}{N}\sum_{j=1}^{N}\left|\mathbf{1}_{C(X_s^i)}(X_s^j - X_s^i) - \mathbf{1}_{C(X_s^i)}(Y_s^j - Y_s^i)\right|$$

$$\le \frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{\partial^{|X_s^j - Y_s^j| + |X_s^i - Y_s^i|}C(X_s^i)}(Y_s^j - Y_s^i)$$

$$\le \frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{\partial^{2\max_{k=1,\cdots,N}|X_s^k - Y_s^k|}C(X_s^i)}(Y_s^j - Y_s^i)$$

$$= \int_{\mathcal{O}} \mathbf{1}_{\partial^{2A_s}C(X_s^i)}(y - Y_s^i)\,\nu_s^N(dy)$$

$$= \int_{\mathcal{O}} \mathbf{1}_{\partial^{2A_s}C(X_s^i)}(y - Y_s^i)\,\rho_s(dy) + \overbrace{\left|\int_{\mathcal{O}} \mathbf{1}_{\partial^{2A_s}C(X_s^i)}(y - Y_s^i)\,(\rho_s - \nu_s^N)(dy)\right|}^{=J_s^{i,N}}$$

$$\le \|\rho_s\|_{L^\infty}\, Vol(\partial^{2A_s}C(X_s^i)) + J_s^{i,N} \le C\|\rho_s\|_{L^\infty}A_s + J_s^{i,N}$$

$$
\begin{aligned}
&\frac{1}{N}\sum_{j=1}^{N}\left|\mathbf{1}_{C(X_s^i)}(X_s^j - X_s^i) - \mathbf{1}_{C(X_s^i)}(Y_s^j - Y_s^i)\right| \\
&\quad \le \frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{\partial^{|X_s^j - Y_s^j| + |X_s^i - Y_s^i|} C(X_s^i)}(Y_s^j - Y_s^i) \\
&\quad \le \frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{\partial^{2\max_{k=1,\cdots,N}|X_s^k - Y_s^k|} C(X_s^i)}(Y_s^j - Y_s^i) \\
&\quad = \int_{\mathcal{O}} \mathbf{1}_{\partial^{2A_s} C(X_s^i)}(y - Y_s^i)\, \nu_s^N(dy) \\
&\quad = \int_{\mathcal{O}} \mathbf{1}_{\partial^{2A_s} C(X_s^i)}(y - Y_s^i)\, \rho_s(dy) + \overbrace{\left|\int_{\mathcal{O}} \mathbf{1}_{\partial^{2A_s} C(X_s^i)}(y - Y_s^i)\, (\rho_s - \nu_s^N)(dy)\right|}^{=J_s^{i,N}} \\
&\quad \le \|\rho_s\|_{L^\infty} Vol(\partial^{2A_s} C(X_s^i)) + J_s^{i,N} \le C\|\rho_s\|_{L^\infty} A_s + J_s^{i,N}
\end{aligned}
$$

$$\frac{1}{N}\sum_{j=1}^{N}\left|\mathbf{1}_{C(X_s^i)}(X_s^j - X_s^i) - \mathbf{1}_{C(X_s^i)}(Y_s^j - Y_s^i)\right|$$

$$\leq \frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{\partial^{|X_s^j - Y_s^j| + |X_s^i - Y_s^i|}C(X_s^i)}(Y_s^j - Y_s^i)$$

$$\leq \frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{\partial^{2\max_{k=1,\cdots,N}|X_s^k - Y_s^k|}C(X_s^i)}(Y_s^j - Y_s^i)$$

$$= \int_{\mathcal{O}}\mathbf{1}_{\partial^{2A_s}C(X_s^i)}(y - Y_s^i)\,\nu_s^N(dy)$$

$$= \int_{\mathcal{O}}\mathbf{1}_{\partial^{2A_s}C(X_s^i)}(y - Y_s^i)\,\rho_s(dy) + \underbrace{\left|\int_{\mathcal{O}}\mathbf{1}_{\partial^{2A_s}C(X_s^i)}(y - Y_s^i)\,(\rho_s - \nu_s^N)(dy)\right|}_{=J_s^{i,N}}$$

$$\leq \|\rho_s\|_{L^\infty} Vol(\partial^{2A_s}C(X_s^i)) + J_s^{i,N} \leq C\|\rho_s\|_{L^\infty}A_s + J_s^{i,N}$$

$$\frac{1}{N}\sum_{j=1}^{N}\left|\mathbf{1}_{C(X_s^i)}(X_s^j - X_s^i) - \mathbf{1}_{C(X_s^i)}(Y_s^j - Y_s^i)\right|$$

$$\leq \frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{\partial^{|X_s^j - Y_s^j| + |X_s^i - Y_s^i|}C(X_s^i)}(Y_s^j - Y_s^i)$$

$$\leq \frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{\partial^{2\max_{k=1,\cdots,N}|X_s^k - Y_s^k|}C(X_s^i)}(Y_s^j - Y_s^i)$$

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$$\leq \|\rho_s\|_{L^\infty} Vol(\partial^{2A_s}C(X_s^i)) + J_s^{i,N} \leq C\|\rho_s\|_{L^\infty}A_s + J_s^{i,N}$$

$$\frac{1}{N}\sum_{j=1}^{N}\left|\mathbf{1}_{C(X_s^i)}(X_s^j - X_s^i) - \mathbf{1}_{C(X_s^i)}(Y_s^j - Y_s^i)\right|$$

$$\leq \frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{\partial^{|X_s^j - Y_s^j| + |X_s^i - Y_s^i|}C(X_s^i)}(Y_s^j - Y_s^i)$$

$$\leq \frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{\partial^{2\max_{k=1,\cdots,N}|X_s^k - Y_s^k|}C(X_s^i)}(Y_s^j - Y_s^i)$$

$$= \int_{\mathcal{O}}\mathbf{1}_{\partial^{2A_s}C(X_s^i)}(y - Y_s^i)\,\nu_s^N(dy)$$

$$= \int_{\mathcal{O}}\mathbf{1}_{\partial^{2A_s}C(X_s^i)}(y - Y_s^i)\,\rho_s(dy) + \overbrace{\left|\int_{\mathcal{O}}\mathbf{1}_{\partial^{2A_s}C(X_s^i)}(y - Y_s^i)\,(\rho_s - \nu_s^N)(dy)\right|}^{=J_s^{i,N}}$$

$$\leq \|\rho_s\|_{L^\infty} Vol\left(\partial^{2A_s}C(X_s^i)\right) + J_s^{i,N} \leq C\|\rho_s\|_{L^\infty}A_s + J_s^{i,N}$$

$$\frac{1}{N}\sum_{j=1}^{N}\left|\mathbf{1}_{C(X_s^i)} - \mathbf{1}_{C(Y_s^i)}\right|(Y_s^j - Y_s^i)$$

$$= \frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{C(X_s^i)\Delta C(Y_s^i)}(Y_s^j - Y_s^i)$$

$$= \int_{\mathcal{O}} \mathbf{1}_{C(X_s^i)\Delta C(Y_s^i)}(y - Y_s^i)\,\nu_s^N(dy)$$

$$= \int_{\mathcal{O}} \mathbf{1}_{C(X_s^i)\Delta C(Y_s^i)}(y - Y_s^i)\,\rho_s(dy) + \tilde{J}_s^{i,N}$$

$$\frac{1}{N}\sum_{j=1}^{N}\left|\mathbf{1}_{C(X_s^i)} - \mathbf{1}_{C(Y_s^i)}\right|(Y_s^j - Y_s^i)$$

$$= \frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{C(X_s^i)\Delta C(Y_s^i)}(Y_s^j - Y_s^i)$$

$$= \int_{\mathcal{O}} \mathbf{1}_{C(X_s^i)\Delta C(Y_s^i)}(y - Y_s^i)\,\nu_s^N(dy)$$

$$= \int_{\mathcal{O}} \mathbf{1}_{C(X_s^i)\Delta C(Y_s^i)}(y - Y_s^i)\,\rho_s(dy) + \tilde{J}_s^{i,N}$$

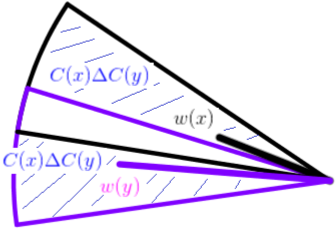
$$
\begin{aligned}
\frac{1}{N}\sum_{j=1}^{N} & \left|\mathbf{1}_{C(X_s^i)} - \mathbf{1}_{C(Y_s^i)}\right|(Y_s^j - Y_s^i) \\
& = \frac{1}{N}\sum_{j=1}^{N} \mathbf{1}_{C(X_s^i)\Delta C(Y_s^i)}(Y_s^j - Y_s^i) \\
& = \int_{\mathcal{O}} \mathbf{1}_{C(X_s^i)\Delta C(Y_s^i)}(y - Y_s^i)\, \nu_s^N(dy) \\
& = \int_{\mathcal{O}} \mathbf{1}_{C(X_s^i)\Delta C(Y_s^i)}(y - Y_s^i)\, \rho_s(dy) + \tilde{J}_s^{i,N}
\end{aligned}
$$

$$\frac{1}{N}\sum_{j=1}^{N}\left|\mathbf{1}_{C(X_s^i)}-\mathbf{1}_{C(Y_s^i)}\right|(Y_s^j-Y_s^i)$$

$$=\frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{C(X_s^i)\Delta C(Y_s^i)}(Y_s^j-Y_s^i)$$

$$=\int_{\mathcal{O}}\mathbf{1}_{C(X_s^i)\Delta C(Y_s^i)}(y-Y_s^i)\,\nu_s^N(dy)$$

$$=\int_{\mathcal{O}}\mathbf{1}_{C(X_s^i)\Delta C(Y_s^i)}(y-Y_s^i)\,\rho_s(dy)+\tilde{J}_s^{i,N}$$



Assume $|w(x)| \geq w_*, \forall x \in \mathbb{R}^2$ then

$$Vol\big(C(x)\Delta C(y)\big) = 2\pi r^2 \frac{|(w(x), w(y))|}{2\pi} \leq \frac{r^2 \|w\|_{Lip}}{w_*}|x-y|,$$

$$\text{if} \quad |(w(x), w(y))| \leq \alpha$$

Then

$$\int_{\mathcal{O}} \mathbf{1}_{C(X_s^i)\Delta C(Y_s^i)}(y - Y_s^i)\, \rho_s(dy) \leq C\|\rho_s\|_{L^\infty}|X_s^i - Y_s^i|$$

Gronwall's estimate

Putting all the estimates together leads to

$$\mathbb{E}\Big[\sup_{i=1,\cdots,N} |X_t^i - Y_t^i|\Big] \le \int_0^t C\|\rho_s\|_{L^\infty}\mathbb{E}\Big[\sup_{i=1,\cdots,N} |X_s^i - Y_s^i|\Big]\, ds$$
$$+\,\mathbb{E}\Big[\sup_{i=1,\cdots,N}\big(J_s^{i,N} + \tilde{J}_s^{i,N} + H_s^{i,N}\big)\Big]\, ds,$$

and then using Grnowall's inequality we have

$$\mathbb{E}\Big[\sup_{i=1,\cdots,N} |X_t^i - Y_t^i|\Big] \le e^{C\int_0^t \|\rho_s\|_{L^\infty}\, ds}\int_0^t \mathbb{E}\Big[\sup_{i=1,\cdots,N}\big(J_s^{i,N} + \tilde{J}_s^{i,N} + H_s^{i,N}\big)\Big]\, ds.$$

It is now a question to get some convergence rate for the quantities

$$\mathbb{E}\Big[\sup_{i=1,\cdots,N} J_s^{i,N}\Big], \quad \mathbb{E}\Big[\sup_{i=1,\cdots,N} \tilde{J}_s^{i,N}\Big], \quad \mathbb{E}\Big[\sup_{i=1,\cdots,N} H_s^{i,N}\Big].$$

Fluctuation term

Recall that

$$J_s^{i,N} = \left|\int_{\mathcal{O}} \mathbf{1}_{\partial^{2\sup_{j=1,\cdots,N}|X_s^j - Y_s^j|} B_r}(y - Y_s^i)\,(\rho_s - \nu_s^N)(dy)\right|$$
$$\leq \sup_{u \geq 0}\left|\int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y - Y_s^i)\,(\rho_s - \nu_s^N)(dy)\right|,$$

where we have replaced the vision cone by the ball of radius r for the sake of simplicity. Now let be $(Y_n)_{n\geq 1}$ a sequence of i.i.d. r. v. of law $\rho \in \mathcal{P}(\mathbb{R}^d)$, K_N some Poisson r. v. of parameter N and define

$$\nu_N := \frac{1}{N}\sum_{i=1}^{N} \delta_{Y_i}, \quad \text{and} \quad M_N := \sum_{i=1}^{K_N} \delta_{Y_i}.$$

M_N is then a Poisson Random Measure of intensity $N\rho$.

Fluctuation term

$$\sup_{u\geq 0}\Big|\int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y-Y_i)\,(\rho-\nu_N)(dy)\Big|$$

$$\leq \sup_{u\geq 0}\Big|\int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y-Y_i)\,(M_N-N\rho)(dy)\Big|N^{-1}$$

$$+\sup_{u\geq 0}\Big|\int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y-Y_i)\,(\nu_N-\frac{M_N}{N})(dy)\Big|$$

$$\leq \sup_{u\geq 0}\Big|\int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y-Y_i)\,\overline{M}_N(dy)\Big|N^{-1}+\|\nu_N-\frac{M_N}{N}\|_{TV}$$

$$\leq N^{-1}\sup_{u\geq 0}|\mathcal{M}_u^{i,N}|+\frac{|K_N-N|}{N},$$

where $\bar{M}$ is the so called compensated measure and

$$\mathcal{M}_u^{i,N}:=\int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y-Y_i)\,\overline{M}_N(dy).$$

Fluctuation term

$$
\begin{aligned}
&\sup_{u\geq 0}\Big|\int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y-Y_i)\,(\rho-\nu_N)(dy)\Big| \\
&\quad \leq \sup_{u\geq 0}\Big|\int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y-Y_i)\,(M_N-N\rho)(dy)\Big|N^{-1} \\
&\quad + \sup_{u\geq 0}\Big|\int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y-Y_i)\,(\nu_N-\frac{M_N}{N})(dy)\Big| \\
&\quad \leq \sup_{u\geq 0}\Big|\int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y-Y_i)\,\overline{M}_N(dy)\Big|N^{-1} + \|\nu_N-\frac{M_N}{N}\|_{TV} \\
&\quad \leq N^{-1}\sup_{u\geq 0}\big|\mathcal{M}_u^{i,N}\big| + \frac{|K_N-N|}{N},
\end{aligned}
$$

where $\bar{M}$ is the so called compensated measure and

$$
\mathcal{M}_u^{i,N} := \int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y-Y_i)\,\overline{M}_N(dy).
$$

Fluctuation term

$$
\begin{aligned}
&\sup_{u\geq 0}\Big|\int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y-Y_i)\,(\rho-\nu_N)(dy)\Big| \\
&\quad\leq \sup_{u\geq 0}\Big|\int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y-Y_i)\,(M_N-N\rho)(dy)\Big|N^{-1} \\
&\quad+\sup_{u\geq 0}\Big|\int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y-Y_i)\,(\nu_N-\frac{M_N}{N})(dy)\Big| \\
&\quad\leq \sup_{u\geq 0}\Big|\int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y-Y_i)\,\overline{M}_N(dy)\Big|N^{-1}+\|\nu_N-\frac{M_N}{N}\|_{TV} \\
&\quad\leq N^{-1}\sup_{u\geq 0}\big|\mathcal{M}_u^{i,N}\big|+\frac{|K_N-N|}{N},
\end{aligned}
$$

where $\bar{M}$ is the so called compensated measure and

$$
\mathcal{M}_u^{i,N} := \int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y-Y_i)\,\overline{M}_N(dy).
$$

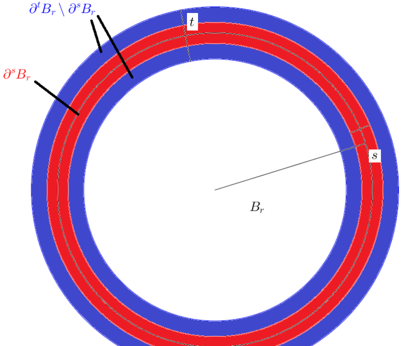
Fluctuation term

Introduce the filtration

$$\mathcal{F}_t^{i,N} := \sigma\{\overline{M}_N(\partial^u B_r + Y_i)\,,\, u \leq t\},$$

then $(\mathcal{M}_u^{i,N})_{u\geq 0}$ is a martingale w. r. t. this filtration. Indeed for $s < t$

$$\begin{aligned}\mathbb{E}\big[\mathcal{M}_t^{i,N}\,|\,\mathcal{F}_s^{i,N}\big] &= \mathbb{E}\Big[\int \mathbf{1}_{\partial^t B_r \setminus \partial^s B_r}(y - Y_i)\,\overline{M}_N(dy)\,|\,\mathcal{F}_s^{i,N}\Big] + \mathbb{E}\big[\mathcal{M}_s^{i,N}\,|\,\mathcal{F}_s^{i,N}\big] \\ &= \mathcal{M}_s^{i,N}.\end{aligned}$$



Fluctuation term

Finally

$$\sup_{i=1,\cdots,N} J^{i,N} = \sup_{i=1,\cdots,N} \sup_{u\geq 0} \Big| \int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y - Y_i)\,(\rho - \nu_N)(dy) \Big|$$

$$\leq \frac{|K_N - N|}{N} + N^{-1} \Big(\sum_{i=1}^{N} \sup_{u\geq 0} |\mathcal{M}_u^{i,N}|^{2m} \Big)^{1/2m},$$

taking the expectation leads to

$$\mathbb{E}\Big[\sup_{i=1,\cdots,N} J^{i,N}\Big] \leq \mathbb{E}\Big[\frac{|K_N - N|}{N}\Big] + N^{-1} \Big(\sum_{i=1}^{N} \mathbb{E}\Big[\sup_{u\geq 0} |\mathcal{M}_u^{i,N}|^{2m}\Big] \Big)^{1/2m}$$

$$\leq N^{-1}\Big(\mathbb{E}\big[|K_N - N|\big] + \Big(\sum_{i=1}^{N} C_m \mathbb{E}\big[|\mathcal{M}_\infty^{i,N}|^{2m}\big] \Big)^{1/2m} \Big)$$

$$\leq N^{-1}\Big(\sqrt{N} + \Big(C_m N \mathbb{E}\big[|\overline{M}_N(\mathbb{R}^d)|^{2m}\big] \Big)^{1/2m} \Big)$$

$$\leq N^{-1}\Big(\sqrt{N} + \Big(C_m N \mathbb{E}\big[|K_N - N|^{2m}\big] \Big)^{1/2m} \Big) \leq C_m N^{-\frac{1}{2} + \frac{1}{2m}},$$

Fluctuation term

Finally

$$\sup_{i=1,\cdots,N} J^{i,N} = \sup_{i=1,\cdots,N} \sup_{u\geq 0} \Big| \int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y - Y_i)\,(\rho - \nu_N)(dy) \Big|$$
$$\leq \frac{|K_N - N|}{N} + N^{-1} \Big(\sum_{i=1}^{N} \sup_{u\geq 0} \big|\mathcal{M}_u^{i,N}\big|^{2m} \Big)^{1/2m},$$

taking the expectation leads to

$$\mathbb{E}\Big[\sup_{i=1,\cdots,N} J^{i,N}\Big] \leq \mathbb{E}\Big[\frac{|K_N - N|}{N}\Big] + N^{-1} \Big(\sum_{i=1}^{N} \mathbb{E}\Big[\sup_{u\geq 0} \big|\mathcal{M}_u^{i,N}\big|^{2m}\Big] \Big)^{1/2m}$$
$$\leq N^{-1} \Big(\mathbb{E}\big[|K_N - N|\big] + \Big(\sum_{i=1}^{N} C_m \mathbb{E}\big[\big|\mathcal{M}_\infty^{i,N}\big|^{2m}\big] \Big)^{1/2m} \Big)$$
$$\leq N^{-1} \Big(\sqrt{N} + \big(C_m N \mathbb{E}\big[|\overline{M}_N(\mathbb{R}^d)|^{2m}\big] \big)^{1/2m} \Big)$$
$$\leq N^{-1} \Big(\sqrt{N} + \big(C_m N \mathbb{E}\big[|K_N - N|^{2m}\big] \big)^{1/2m} \Big) \leq C_m N^{-\frac{1}{2} + \frac{1}{2m}},$$

Fluctuation term

Finally

$$\sup_{i=1,\cdots,N} J^{i,N} = \sup_{i=1,\cdots,N} \sup_{u\geq 0} \Big| \int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y - Y_i)\,(\rho - \nu_N)(dy) \Big|$$

$$\leq \frac{|K_N - N|}{N} + N^{-1} \Big(\sum_{i=1}^{N} \sup_{u\geq 0} |\mathcal{M}_u^{i,N}|^{2m} \Big)^{1/2m},$$

taking the expectation leads to

$$\mathbb{E}\Big[\sup_{i=1,\cdots,N} J^{i,N}\Big] \leq \mathbb{E}\Big[\frac{|K_N - N|}{N}\Big] + N^{-1}\Big(\sum_{i=1}^{N} \mathbb{E}\big[\sup_{u\geq 0} |\mathcal{M}_u^{i,N}|^{2m}\big]\Big)^{1/2m}$$

$$\leq N^{-1}\Big(\mathbb{E}\big[|K_N - N|\big] + \Big(\sum_{i=1}^{N} C_m \mathbb{E}\big[|\mathcal{M}_\infty^{i,N}|^{2m}\big]\Big)^{1/2m}\Big)$$

$$\leq N^{-1}\Big(\sqrt{N} + \big(C_m N \mathbb{E}\big[|\overline{M}_N(\mathbb{R}^d)|^{2m}\big]\big)^{1/2m}\Big)$$

$$\leq N^{-1}\Big(\sqrt{N} + \big(C_m N \mathbb{E}\big[|K_N - N|^{2m}\big]\big)^{1/2m}\Big) \leq C_m N^{-\frac{1}{2} + \frac{1}{2m}},$$

Fluctuation term

Finally

$$\sup_{i=1,\cdots,N} J^{i,N} = \sup_{i=1,\cdots,N} \sup_{u\geq 0} \Big| \int_{\mathcal{O}} \mathbf{1}_{\partial^u B_r}(y - Y_i)\,(\rho - \nu_N)(dy) \Big|$$
$$\leq \frac{|K_N - N|}{N} + N^{-1} \Big(\sum_{i=1}^{N} \sup_{u\geq 0} |\mathcal{M}_u^{i,N}|^{2m} \Big)^{1/2m},$$

taking the expectation leads to

$$\mathbb{E}\Big[\sup_{i=1,\cdots,N} J^{i,N} \Big] \leq \mathbb{E}\Big[\frac{|K_N - N|}{N} \Big] + N^{-1} \Big(\sum_{i=1}^{N} \mathbb{E}\Big[\sup_{u\geq 0} |\mathcal{M}_u^{i,N}|^{2m} \Big] \Big)^{1/2m}$$
$$\leq N^{-1} \Big(\mathbb{E}\big[|K_N - N|\big] + \Big(\sum_{i=1}^{N} C_m \mathbb{E}\big[|\mathcal{M}_\infty^{i,N}|^{2m} \big] \Big)^{1/2m} \Big)$$
$$\leq N^{-1} \Big(\sqrt{N} + \big(C_m N \mathbb{E}\big[|\overline{M}_N(\mathbb{R}^d)|^{2m} \big] \big)^{1/2m} \Big)$$
$$\leq N^{-1} \Big(\sqrt{N} + \big(C_m N \mathbb{E}\big[|K_N - N|^{2m} \big] \big)^{1/2m} \Big) \leq C_m N^{-\frac{1}{2} + \frac{1}{2m}},$$

Conclusion

Cucker-Smale with vision cone

Particle system

$$\begin{cases} \dfrac{dX_t^i}{dt} = V_t^i \\ \dfrac{dV_t^i}{dt} = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \mathbf{1}_{C(V_t^i)}(X_t^i - X_t^j)(V_t^j - V_t^i), \end{cases}$$

Nonlinear limit

$$\begin{cases} \partial_t f_t + v \cdot \nabla_x f_t + \nabla_v \cdot (f_t F(f_t)) = 0, \\ F(f_t)(x,v) = \displaystyle\int_{\mathbb{R}^d} \mathbf{1}_{C(v)}(x-z)(w-v) f_t(dz, dw). \end{cases}$$

- One main feature of this model is to keep velocity bounded by the maximal velocity at initial time.
- We would like to add a diffusion in velocity, but it would destroy this bound.
- Confining a position in a bounded set makes more physical sense than confining a velocity.

The End

Thank you for your attention!