The Cauchy problem for the quantum Boltzmann equation for bosons at very low temperature

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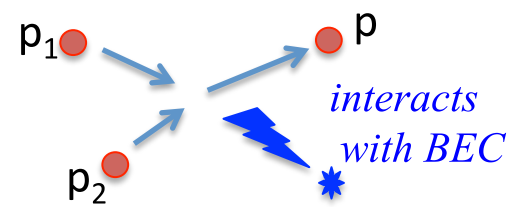
Bose-Einstein Condensates (BEC) and cold bosonic gases below condensation temperature are well described by the evolution of the homogeneous evolution $f := f(t,p), p \in \mathbb{R}^3$ momenta space. Uehling&Uhlenbeck'33; Kirkpatrick&Dorfmann'80s;Eckrn'84; Zeremba,Griffin&Nikumi'99; Zoller'08



$$\frac{\mathrm{d}f}{\mathrm{d}t} = n_c(t)\, Q[f], \qquad f(0,\cdot) = f_0,$$

$$Q[f] := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{d}p_1 \mathrm{d}p_2 [R(p,p_1,p_2) - R(p_1,p,p_2) - R(p_2,p_1,p)],$$

$$\boxed{\begin{array}{c} p=p_1+p_2 \\ \omega(p)=\omega(p)_1+\omega(p_2) \end{array}}$$

with $R(p,p_1,p_2) := \mathcal{M}(p,p_1,p_2)|^2 \left[\delta\left(\frac{\omega(p)}{k_B T} - \frac{\omega(p_1)}{k_B T} - \frac{\omega(p_2)}{k_B T}\right)\delta(p - p_1 - p_2)\right] \times$

$$[f(p_1)f(p_2)(1+f(p)) - (1+f(p_1)(1+f(p_2))f(p)].$$

- $\beta := k_B T^{-1} > 0$ for k_B Boltzmann constant and T temperature of quasiparticles at equilibrium.
- $\mathcal{M}(p,p_1,p_2)|^2$ is the transition probability rate.
- and the particle energy $\omega(p)$ is given by the Bogoliubov dispersion law:

$$\omega(p) = \left[\frac{g n_c}{m}|p|^2 + \left(\frac{|p|^2}{2m}\right)^2\right]^{1/2},$$

where m is the mass of the particles, g is the interaction coupling constant, $n_c = n_c(t) = |\Psi|^2(t)$ is the **density of particles in the BEC.**

Note that mass, $\int_{\mathbb{R}^3} Q[f](p)dp = m_0(t)$ is not a conserved quantity.

• The quantum kinetic equation couples with the condensate density distribution $n_c(t)$ associated to the wave function $\Psi(x,t)$, under some further assumptions, yielding the differential system Arkeryd & Nouri '12, Spohn'10, Pomeau et al

$$\begin{cases} \frac{\mathrm{d}f}{\mathrm{d}t} & = \; Q_4[f] \; + \; n_c(t)\, Q_3[f] \\ f(0,\cdot) & = f_0\,, \end{cases}$$

Q_4 Boltzmann Nordheim (BN) eq.

Q_3 the QBE-LT eq.

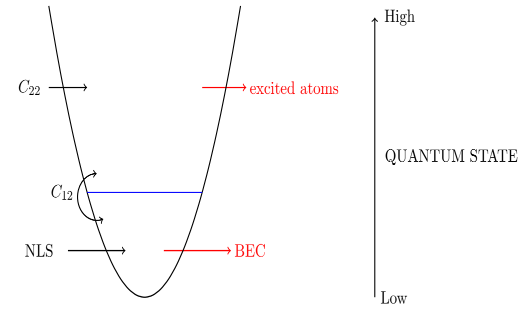


Figure 1: The Bose-Einstein Condensate (BEC) and the excitations.

and the condensate density distribution evolves according to

$$\begin{cases} \frac{\mathrm{d}n_c}{\mathrm{d}t} & = -n_c \int_{\mathbb{R}^3} Q_3[f]dp, \\ n_c(0) & = n_0\,, \end{cases} \qquad \text{or, equivalently} \qquad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\log n_c & = -\int_{\mathbb{R}^3} Q_3[f]dp, \\ \log(n_c(0)) & = \log(n_0). \end{cases}$$

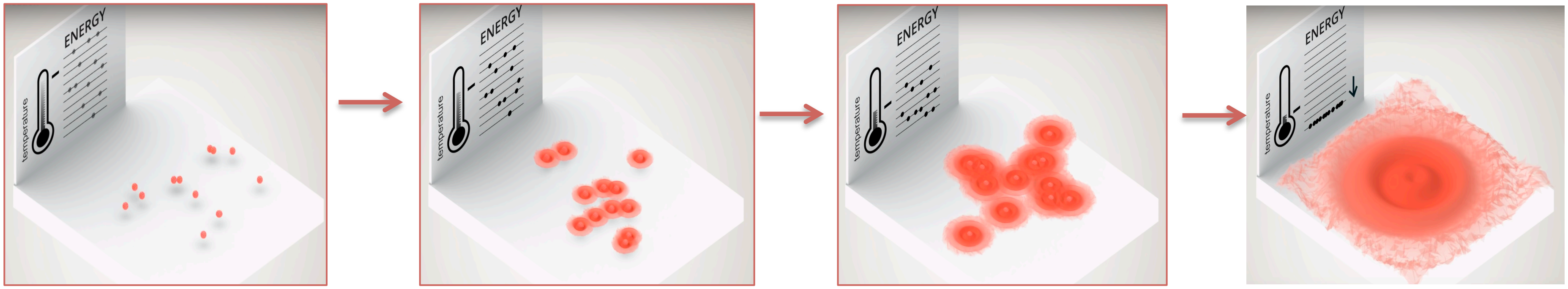
• In the **post-nucleation of the condensate** regime given by times $t > t_*$, for t_* the finite time blow up of f solution of the kinetic equation (Escobedo&Velazquez'06&'15, Sphon'10), the **BN collision operator** $Q_4[f](p,t)$ **is of lower order** and the kinetic dynamics are regulated by the evolution of

$$\frac{\mathrm{d}f}{\mathrm{d}t} = n_c(t)\, Q_3[f] \qquad \text{for initial data} \quad f(0,\cdot) \;=\; f_0\,.$$

• We only focus on the QBE-LT and so we impose $\quad n_c \in C^1[0,\infty)$,

and there exists constants $\underline{n}_c, \overline{n}_c > 0$ such that $\underline{n}_c < n_c(t) < \overline{n}_c, \quad \forall t \in [0,\infty)$.

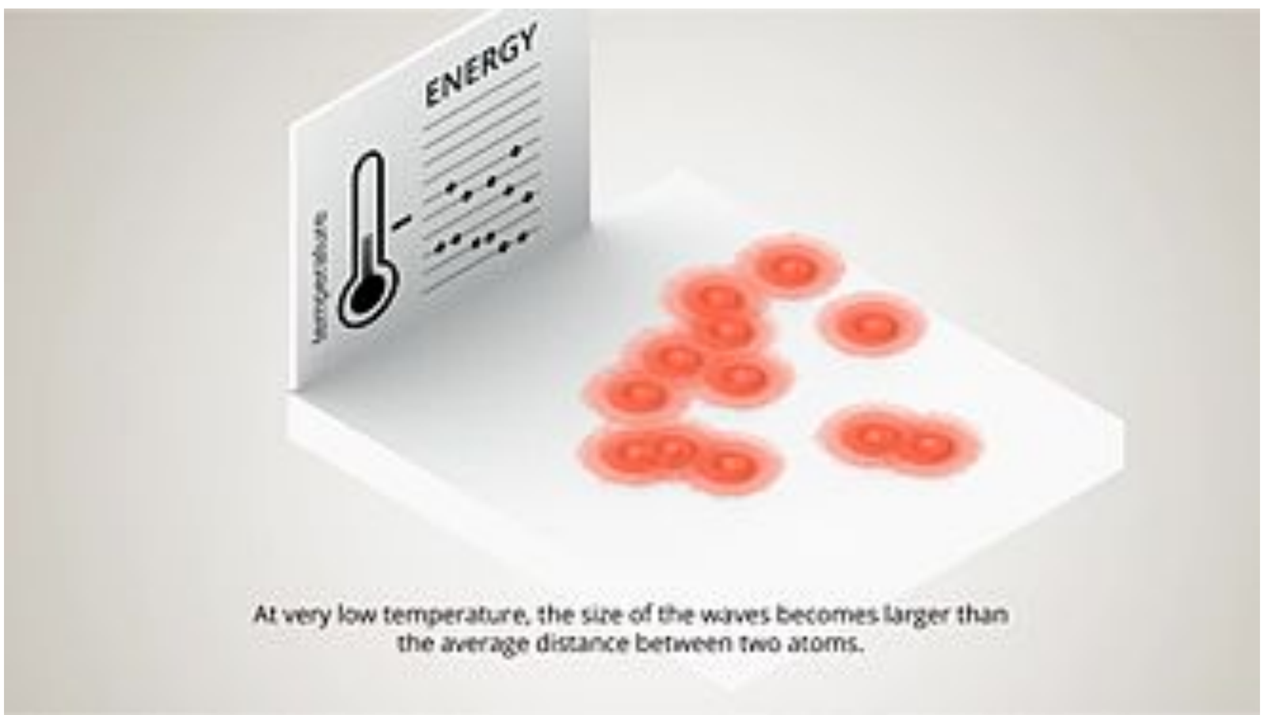
This assumption is physically meaningful: It says that the condensate does not vanish, and its density distribution is uniformly bounded from above and below in time.



High energy

temperature

Low energy (cold)



The corresponding **equilibrium distribution** f_∞ **of the collisional equation** has the form

$$\boxed{f_\infty(p) = \frac{1}{e^{\beta\omega(p)} - 1}\,,}$$

for $\beta = (k_B T)^{-1}$, as is usually referred as a Bose-Einstein distribution, i.e. solutions are not expected to be in $L^\infty(\mathbb{R}^3)$

• Restricting the range of the temperature T such that $k_B T \ll 1$ i.e. a *cold gas regime*, while $(gn_c/m)^{1/2}(k_B T)^{-1} = O(1)$, for the condensate density n_c and interaction g,

then the Bogoliubov dispersion law becomes

$$\boxed{\frac{1}{k_B T}\left[\frac{gn_c}{m}|p|^2 + \left(\frac{|p|^2}{2m}\right)^2\right]^{1/2} \approx \frac{c}{k_B T}|p|, \quad \text{where } \ c := \sqrt{\frac{gn_c}{m}}\,,}$$

In particular, the energy is defined by the classical phonon dispersion law (still using the same notation), see

$$\boxed{\omega(p) = c|p|,}$$

Bogoliubov dispersion relation for a cold gas regime

Under this *cold gas regime*, the transition probability $\mathcal{M}$ rate is approx. by

$$|\mathcal{M}|^2 = \kappa\,|p||p_1||p_2|\,, \qquad \text{with} \quad \kappa > 0$$

The approximation holds at low temperature where only low momentum excitations are relevant.

The binary quantum collisional form for cold temperature BECs: W.l.o.g., we assume $c(k_B T)^{-1} = 1$, with $T \ll 1$, in the reduced phonon dispersion law, then cubic quantum collisional integral for becomes quadratic

$$Q[f] := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathrm{d}p_1 \mathrm{d}p_2 [R(p, p_1, p_2) - R(p_1, p, p_2) - R(p_2, p_1, p)] \,,$$

$$R(p, p_1, p_2) := |p||p_1||p_2| [\delta\,(|p| - |p_1| - |p_2|)\, \delta(p - p_1 - p_2)] \times$$

$$[f(p_1) f(p_2)(1 + f(p)) - (1 + f(p_1)(1 + f(p_2)) f(p)] \,.$$

and we will rigorously show that it can be split into the difference of two positive quadratic operators forms *gain* and *loss*

$$Q[f](t, p) = Q^+[f](t, p) - f(t, p)\, \nu[f](t, p) \,.$$

where $\nu[f](t, p)$ is the collision or interaction frequency.

The binary quantum collisional form for cold temperature BECs The gain operator is defined by

$$Q^{+}[f](t,p) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{d}p_1 \mathrm{d}p_2 |p||p_1||p_2| \delta(p - p_1 - p_2) \\ \times \delta(|p| - |p_1| - |p_2|) f(t,p_1) f(t,p_2) + 2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{d}p_1 \mathrm{d}p_2 |p||p_1||p_2| \\ \times \delta(p_1 - p - p_2) \delta(|p_1| - |p| - |p_2|) [2 f(t,p) f(t,p_1) + f(t,p_1)] \,.$$

The loss operator is local in $f(t,p)$: $Q^{-}[f] := f\,\nu[f]$, with a nonlocal operator $\nu[f](t,p)$ the *collision frequency or attenuation coefficient*, is defined by

$$\nu[f](t,p) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{d}p_1 \mathrm{d}p_2 |p||p_1||p_2| \delta(p - p_1 - p_2) \\ \times \delta(|p| - |p_1| - |p_2|) [2 f(t,p_1) + 1] + 2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{d}p_1 \mathrm{d}p_2 |p||p_1||p_2| \\ \times \delta(p_1 - p - p_2) \delta(|p_1| - |p| - |p_2|) f(t,p_2) \,,$$

In order to grant the split of the collision operator in gain and loss parts, it is necessary that $\nu[f](t,p)$ is well defined.

This equation was study by Spohn'06 in the torus, and Arkeryd and Nouri'13 for cut-off transition probability $\mathcal{M}$ forms.

Properties of the Quantum Boltzmann Eq. for Bosons at Low Temperature (QBE-LT)

Proposition (**WeakFormulation**) For any suitable test function φ,

$$\boxed{\int_{\mathbb{R}^3} dp\, Q[f](p)\varphi(p)} = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} dp\, dp_1\, dp_2\; |p||p_1||p_2|\delta(p - p_1 - p_2)$$

$$\times\, \delta(|p| - |p_1| - |p_2|)\Big[f(p_1)f(p_2) - f(p_1)f(p) - f(p_2)f(p) - f(p)\Big]$$

$$\times\Big[\varphi(p) - \varphi(p_1) - \varphi(p_2)\Big]$$

$$\boxed{\begin{aligned} &= 2\pi \int_{\mathbb{R}^3} dp_1 \int_{\mathbb{R}^+} d|p_2|\, n_c \left|p_1 + |p_2|\widehat{p_1}\right||p_1||p_2|^3\Big[f(p_1)f(|p_2|\widehat{p_1})\\ &- f(p_1)f(p_1 + |p_2|\widehat{p_1}) - f(|p_2|\widehat{p_1})f(p_1 + |p_2|\widehat{p_1}) - f(p_1 + |p_2|\widehat{p_1})\Big]\times\\ &\Big[\varphi(p_1 + |p_2|\widehat{p_1}) - \varphi(p_1) - \varphi(|p_2|\widehat{p_1})\Big]\end{aligned}}$$

urthermore, for **radially symmetric functions** $f(p) := f(|p|)$ and $\varphi(p) := \varphi(|p|)$,

$$\boxed{\begin{aligned}\int_{\mathbb{R}^3} dp\, Q[f](p)\varphi(p) &= 8\pi^2 \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} d|p_1|\, d|p_2|\, (|p_1| + |p_2|)\, |p_1|^3|p_2|^3 \times\\ &\Big[f(|p_1|)f(|p_2|) - f(|p_1|)f(|p_1| + |p_2|) - f(|p_2|)f(|p_1| + |p_2|)\\ &- f(|p_1| + |p_2|)\Big] \times \Big[\varphi(|p_1| + |p_2|) - \varphi(|p_1|) - \varphi(|p_2|)\Big].\end{aligned}}$$

Properties of the Quantum Boltzmann Eq. for Bosons at Low Temperature (QBE-LT)

Conservation of momentum and energy properties

$p = p_1 + p_2$ and $|p| = |p_1| + |p_2|$. Testing with $\varphi(p) = p$ and $|p|$:

$$\frac{d}{dt}\int_{R^3} f(p)\begin{pmatrix} p \\ |p| \end{pmatrix} dp = \int_{\mathbb{R}^9} [f(p_1)f(p_2) - (f(p_1) + f(p_2) + 1)\, f(p)\,]\delta(p - p_1 - p_2)$$

$$\delta(|p| - |p_1| - |p_2|)|p||p_1||p_2|\begin{pmatrix} p & -p_1 & -p_2 \\ |p| & -|p_1| & -|p_2| \end{pmatrix} dp\, dp_1\, dp_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

• $f(p,t)$ is a density function of a *thermal cloud* where atoms could move in & out of the condensate, so **the total mass of the system thermal cloud - condensate is unchanged**, but the mass of each component is not conserved.

That means the coupled QBE average and GP eqs. satisfy that their sum of masses are conserved, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^3} f(t,p)dp \; + \; n_c(t)\right) \; = \; 0, \quad \Longleftrightarrow \quad \begin{cases} \frac{\mathrm{d}n_c}{\mathrm{d}t} & = -n_c \int_{\mathbb{R}^3} Q[f]dp = \frac{\mathrm{d}}{\mathrm{d}t} f(t,p) \\ n_c(0) & = n_0\,, \quad f(0,p) = f_0\,. \end{cases}$$

Quantum H and Boltzmann Theorem for Bosons at low temperature

The Quantum H-Theorem: testing with $\varphi(p) = \log\left(\frac{f(p)}{f(p)+1}\right)$

$$\int_{R^3} \partial_t f(p) \log\left(\frac{f(p)}{f(p)+1}\right) dp = \frac{d}{dt}\int_{R^3} f\log f - (1+f)\log(1+f)dp \leq 0.$$

The Quantum Boltzmann Theorem:

$$\int_{R^3} Q(M)(p)\ \log\left(\frac{M(p)}{M(p)+1}\right) = 0 \iff \frac{M(p_1)}{M(p_1)+1}\,\frac{M(p_2)}{M(p_2)+1} - \frac{M(p)}{M(p)+1} = 0, \iff$$

for $\log\left(\dfrac{M(p)}{M(p)+1}\right) := -\alpha\omega(p)$ for some $\alpha > 0$

$\omega(p_1) + \omega(p_2) = \omega(p_1)$ is conservation of the condensate energy ,

for $\omega(p)$ the associated dispersion relation.

Note that for the low temperature condensate regime $-\alpha\omega(p) = -\alpha|p|$ **, or** $|p_1| + |p_2| - |p| = 0$

Such $M(p) = \frac{1}{\exp(\alpha\omega(p) - 1)}$ **is the symmetric equilibrium distribution called the Bose-Einstein pdf uniquely associated to the initial state of the QBE-LT flow.**

(Escobedo &Tran,2015)

Cauchy problem for the Quantum Boltzmann Equation for Bosons at low temperature

(R.Alonso, IMG, MB.Tran on ArXiv'16)

We study the **Cauchy problem** and **high energy tail behavior for the QBE-LT,** context of radially symmetric solutions.

Inspired by recent techniques developed for the classical Boltzmann equations for hard potentials IMG, Panferov&Villani'06-09; Alonso et al,'12, Taskovic, et al'15 ,

- **i-** Obtain key a priori estimates on the moments of QBE-LT equation
- **ii-** from estimates in (**i-**) we show the creation and propagation of polynomial moments,
- **iii-** Using the estimates in (**ii-**) and results for ODE flows in Banach spaces, we found natural conditions to show existence and uniqueness of solutions of radially symmetric solutions to the initial value problem.
  - **Existence** is based on a **Hölder estimate** and a **sub-tangent condition** type for Q,
  - **Uniqueness** is based on a **one-side Lipschitz estimate**.
- **iv-** The solutions constructed in (**iii-**), **propagate and generate of Mittag-Leffler moments**: i.e. the $f(t,p)$ solution has exponential high energy tails in p-space in the sense of $L^1(\mathbb{R}^3)$.

Cauchy problem for the QBE-LT - Part ii- (R.Alonso, IMG, MB.Tran on ArXiv'16)

A priori estimates: For $f(t,p) = f(t,|p|)$ any radial solution of the QBE-LT in the Banach space

$$L^1(\mathbb{R}^3, |p|^k \mathrm{d}p) := \Big\{ f \text{ measurable} \;\Big|\; \int_{\mathbb{R}^3} \mathrm{d}p \, |f(p)||p|^k < \infty, \; k \geq 1 \Big\}.$$

with time continuity for k sufficiently large.

Moments of order k on $\mathbb{R}^3$: $\mathcal{M}_k\langle f\rangle(t) := \int_{\mathbb{R}^3} dp\, f(t,|p|)|p|^k$.

And using spherical integrations, (dp on $\mathbb{R}^3$ is reduced to d$|p|$ on $\mathbb{R}_+$) to define

Line Moments of order k: on $\mathbb{R}_+$ $m_k\langle f\rangle(t) := \int_0^\infty d|p|\, f(t,|p|)|p|^k$.

Note: $\mathcal{M}_k\langle f\rangle(t)$ and $m_{k+2}\langle f\rangle(t)$ are equivalent for radially symmetric functions.

Thus, given the Low Temperature dispersion law $\omega(p) = |p|$, **conservation of energy** means

$\mathcal{M}_1\langle f\rangle(t) = \mathcal{M}_1\langle f_0\rangle$, or equivalently $m_3\langle f\rangle(t) = m_3\langle f_0\rangle$ for radially sym. solutions

Cauchy problem for the QBE-LT - Part i- (R.Alonso, IMG, MB.Tran on ArXiv'16)

- **Proposition: (Line-Moment Differential Inequalities)** For $1 \leq k\gamma \leq k$ and some universal constants C_1 and C_2, the line-moment $m_k = m_k(t)$ satisfies the Ordinary differential Inequality

$$\boxed{\frac{d}{dt} m_{k\gamma+2}(t) \leq C_1 \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} (m_{i\gamma+4} m_{3+(k-i)\gamma} + m_{i\gamma+3} m_{4+(k-i)\gamma})(t) - C_2\, m_{k\gamma+8}(t)\,.}$$

- Control the k-line-moment of the **positive part** of the QBE-LT **from above** by forming partial sums of line-moments and apply the **moment interpolation estimates for** $0 < \rho_1 \leq \rho \leq \rho_2$

$$m_\rho \leq m_{\rho_1}^{\gamma} m_{\rho_2}^{1-\gamma}, \qquad \text{with } \rho = \gamma\rho_1 + (1-\gamma)\rho_2 \text{ and } 0 < \gamma < 1\,.$$

- Control the k-line-moment of the **negative part** of the QBE-LT from below by a higher order line-moment:

  $|r_1 + r_2|^{k\gamma} - r_1^{k\gamma} - r_2^{k\gamma} > 0\,,$ **implies the control from below**

$$\int_0^{+\infty} f(t,r)\, \nu[f](t,r)\, dr \geq C \int_0^{+\infty} r^{k\gamma+8} f(t,r) dr = C_2\, m_{k\gamma+8}(t)\,.$$

These estimates do not need bounds on the initial entropy, since the moment estimates are independent of the bounded entropy.

Easier that classical Boltzmann lower estimate for the collision frequency since *there is no angular scattering*

Cauchy problem for the QBE-LT - Part i- (R.Alonso, IMG, MB.Tran on ArXiv'16)

These differential inequalities imply the following fundamental property of regularization in the sense of L^1: **generation or creation of line-moments**

Theorem: (**moments generation**) For $f_0(p) = f_0(|p|)$, $\mathfrak{h}_3 := m_3(0) < \infty$. Then, there exists a constant $C_k(\mathfrak{h}_3)$ such that the following k-line-moment satisfies

$$m_k(t) \leq C_k(\mathfrak{h}_3)\left(1 - e^{-C_k t}\right)^{-\frac{k-3}{6}}, \quad \forall\, k > 3\,.$$

Moreover, if $m_k(0) < \infty$, then $m_k(t) \leq \max\left\{m_k(0), C_k(\mathfrak{h}_3)\right\}.$

Comments on the proof: From the k-line-moments **Ordinary Differential Inequalities** the following nonlinear k-line-moment differential inequality follows

$$\frac{\partial}{\partial t} m_k(t) \;\leq\; C_k(1 + \; m_k(t)) \;-\; C'' m_k^{\frac{k+3}{k-3}}(t)\,.$$

So, comparing with the solution of the Bernoulli equation

$$\frac{\partial}{\partial t} Y(t) \;\leq\; C_k Y(t) \;-\; C'' Y^{\frac{k+3}{k-3}}(t)\,,$$

yields, for $k > 3$, the **global in time bound**

$$Y(t) \leq C_k(\mathfrak{h}_3)(1 - e^{-\frac{6C_k}{k-3}t})^{-\frac{k-3}{6}}, \text{ with } C_k(\mathfrak{h}_3) := (C_k/C'')^{\frac{k-3}{6}} \text{ constant}$$

In addition, $m_k(0) < \infty$ is finite, then the bound may be improved at $t = 0$,

Cauchy problem for the QBE-LT - Part ii- (R.Alonso, IMG, MB.Tran on ArXiv'16) **Existence and Uniqueness Theorem:** (Refs: Martin'83 on Functional Analysis & A. Bressan's unpublished notes, 06, Alonso, Bagland,Chen &Lods,'16)

Let $E := (E, \|\cdot\|)$ be a Banach space, $\mathcal{S}$ be a bounded, convex, closed subset of E, & the operator $Q : \mathcal{S} \to E$ satisfies:

- **Hölder continuity** $\big\|Q[f] - Q[g]\big\| \leq C\|f-g\|^{\beta}, \quad \beta \in (0,1), \ \forall\, f, g \in \mathcal{S}$,

- **Sub-tangent condition:** $\liminf_{h\to 0^+} h^{-1}\mathrm{dist}(f + hQ[f], \mathcal{S}) = 0, \qquad \forall\, f \in \mathcal{S}$,

- **One-sided Lipschitz condition:** $[Q[f] - Q[g], f-g] \leq C\|f-g\|, \ \forall\, f, g \in \mathcal{S}$,

where $[\varphi, \phi] := \lim_{h\to 0^-} h^{-1}(\|\phi + h\varphi\| - \|\phi\|)$.

Then, $\exists\, \mathfrak{h}_{10}(\mathfrak{h}_3)$ such that $\partial_t f(t,|p|) = Q[f](t,|p|)$ on $[0,\infty)\times E$, $f(0) = f_0 \in \mathcal{S}_{\mathfrak{h}_3}$,

has a unique solution in $C^1((0,\infty), E) \cap C([0,\infty), \mathcal{S})$, where the set $\mathcal{S} := \mathcal{S}_{\mathfrak{h}_3}$ is

$$\mathcal{S}_{\mathfrak{h}_3} := \Big\{ f \in L^1(\mathbb{R}^3, |p|dp) \,\Big|\, \mathbf{i}\text{– } f \geq 0 \text{ \& radially symmetric}, \ \mathbf{ii}\text{–} m_3\langle f\rangle = \int_{\mathbb{R}_+} d|p|\, f(|p|)|p|^3 = \mathfrak{h}_3$$

$$\mathbf{iii}\text{–}\ m_{10}\langle f\rangle = \int_{\mathbb{R}_+} d|p|\, f(|p|)|p|^{10} \ \leq\ \mathfrak{h}_{10} = \mathfrak{h}_{10}(\mathfrak{h}_3) \Big\}.$$

Challenge: to show that the generation of moments estimates are enough to show the above 3 properties and $\mathcal{S}_{\mathfrak{h}_3}$ is non-empty.

Cauchy problem for the QBE-LT - Part ii- (R.Alonso, IMG, MB.Tran on ArXiv'16) **Comments on the proof:**

- The constant $\mathfrak{h}_{10}(\mathfrak{h}_3)$ will be determined in the **sub-tangent condition** proof and will satisfy that $\mathfrak{h}_{10} \leq m_{10}\langle f\rangle(\mathfrak{h}_3)$ obtained from to the moments generation estimate.

Lemma: (**Hölder continuity**) The collision operator $Q : \mathcal{S}_{\mathfrak{h}_3} \to L^1(\mathbb{R}^3, |p|\mathrm{d}p)$ is Hölder continuous, with the following estimate

$$m_3\langle |Q[f] - Q[g]|\rangle \leq A_1\, m_3\langle |f-g|\rangle^{\frac{1}{7}} + A_2\, m_3\langle |f-g|\rangle\,,$$

valid for all $f,\ g \in \mathcal{S}_{\mathfrak{h}_3}$. The constants A_i, for $i = \{1,2\}$, depend only on $\mathfrak{h}_3$ and $\mathfrak{h}_{10}(\mathfrak{h}_3)$.

Remark: For any $f \in \mathcal{S}_{\mathfrak{h}_3}$, properties **i.** and **ii.** imply the interpolation estimates for line-moments $m_5\langle f\rangle \leq C_5$ **and** $m_6\langle f\rangle \leq C_6$, with $\gamma = \frac{2}{7}$ and $\gamma = \frac{3}{7}$ being positive constants depending only on $\mathfrak{h}_3$ and $\mathfrak{h}_{10}(\mathfrak{h}_3)$, resp. These are sufficient estimates to control

$$m_3\langle |Q[f]-Q[g]|\rangle = \int_{\mathbb{R}^3} \mathrm{d}p\, \big|Q[f]-Q[g]\big|(p)|p| = \int_{\mathbb{R}^3} \mathrm{d}p\, (Q[f]-Q[g])(p)\,\mathrm{sign}(Q[f]-Q[g])(p)|p|$$

$$= \int_{\mathbb{R}^9} \mathrm{d}p\, \mathrm{d}p_1 \mathrm{d}p_2\, |p\, p_1 p_2| \delta(p-p_1-p_2)\delta(|p|-|p_1|-|p_2|)\times$$

$$\Big[f(p_1)f(p_2) - 2f(p_2)f(p) - f(p) - g(p_1)g(p_2) + 2g(p_2)g(p) + g(p)\Big]\times$$

$$\Big[|p|\mathrm{sign}(Q[f]-Q[g])(p) - |p_1|\mathrm{sign}(Q[f]-Q[g])(p_1) - |p_2|\mathrm{sign}(Q[f]-Q[g])(p_2)\Big]$$

$$\leq \sum_{j=1}^{3} Q_j(m_3\langle |f-g|\rangle)$$

Cauchy problem for the QBE-LT - Part ii- (R.Alonso, IMG, MB.Tran on ArXiv'16) Next crucial estimates on the negative contribution of $Q[f] = Q^{+}[f] - f\,\nu[f]$ are needed:

From above: $$\nu[f](p) = \int_0^{|p|} \mathrm{d}r\,|p| r^3 (|p| - r)[2f(r) + 1] + 2\int_{\mathbb{R}_+} \mathrm{d}r |p| (|p| + r) r^3 f(r)$$
$$\leq C\,|p|\left(m_3\langle f\rangle^{\frac{5}{4}} + m_4\langle f\rangle + |p|^5\right).$$

The **Sub-tangent condition:** follows as a corollary of the following

Proposition: Fix $f \in \mathcal{S}_{\mathfrak{h}_3}$. Then, for any $\epsilon > 0$, there exists $h_* := h_*(f,\epsilon) > 0$, such that the ball centered at $f + hQ[f]$ with radius $h\,\epsilon > 0$ intersects $\mathcal{S}_{\mathfrak{h}_3}$, that is,

$$B(\,f + hQ[f]\,,\, h\epsilon\,) \cap \mathcal{S}_{\mathfrak{h}_3}, \text{ is non-empty for any } 0 < h < h_*.$$

Control $\nu[f]$ from below: Set $f_R(p) := \chi_R(p) f(p)$, then set $w_R := f + hQ[f_R]$ to be control from below, and show we can choose h_1 such that

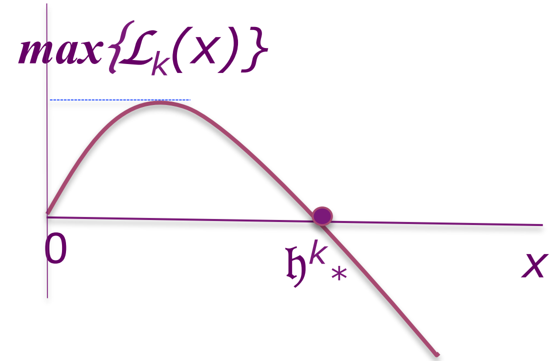
$$w_R = f + Q^{+}[f_R] - h f_R\,\nu[f_R] \geq f - h f_R\,\nu[f_R] \geq f\Big(1 - h\,C(\mathfrak{h}_3, \mathfrak{h}_{10}) R(1 + |R|^5)\Big) \geq 0\,,$$

for any $0 < h < h_1 := 1/\Big(\,C(\mathfrak{h}_3, \mathfrak{h}_{10}) R(1 + |R|^5)\,\Big)$.

Then $\quad w_R := f + hQ[f_R], \quad$ for $\quad f_R(p) := \chi_R(p) f(p),$ satisfies:

- by **conservation of energy** $\int_{\mathbb{R}^3} \mathrm{d}p\, Q[f_R])|p|^3 = 0 \;\Rightarrow\; m_3\langle w_R\rangle = \mathfrak{h}_3$.

- From the **moments generation estimates**, for any $k > 3$,

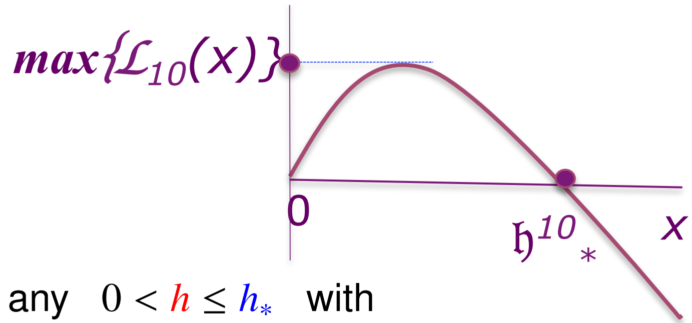


$$\int_{\mathbb{R}^3} dp Q[f]|p|^k \leq \mathcal{L}_k(m_k\langle f\rangle)$$

$$:= C_k \left[m_k\langle f\rangle^{\frac{k-1}{k-3}} + m_k\langle f\rangle^{\frac{k+3}{5(k-3)}} + m_k\langle f\rangle^{\frac{1}{2}} \right] - \frac{C''}{2} m_k\langle f\rangle^{\frac{k+3}{k-3}}$$

Noting the map $\mathcal{L}_k : [0,\infty) \to \mathbb{R}$ **has only one root (denoted by** $\mathfrak{h}_*^k$ **)** at which $\mathcal{L}_k$ changes from positive to negative for any $k > 3 \;\Rightarrow$

$$\int_{\mathbb{R}^3} \mathrm{d}p\, Q[f]|p|^k \;\leq\; \mathcal{L}_k(m_k\langle f\rangle) \;\leq\; \max_{0\leq x\leq \mathfrak{h}_*^k}\{\mathcal{L}_k(x)\}, \qquad f \in \mathcal{S}.$$



• Fix $k = 10$ and define $\quad \mathfrak{h}_{10} := \mathfrak{h}_*^{10} + \max_{0\le x\le \mathfrak{h}_*^{10}}\{\mathcal{L}_{10}(x)\}.$

$\Rightarrow \quad m_{10}\langle w_R\rangle \le \mathfrak{h}_*^{10} + h \max_{0\le x\le \mathfrak{h}_*^{10}}\{\mathcal{L}_{10}(x)\} \le \mathfrak{h}_{10},$ for any $0 < h \le h_*$ with

$$h_* = \min\left\{1, 1/\left(C(\mathfrak{h}_3)R(f)(1+|R(f)|^5)\right)\right\} \Rightarrow \boxed{w_R \in \mathcal{S}_{\mathfrak{h}_3} \text{ uniformly in } R.}$$

• Finally, by the **Hölder continuity property** and taking $R(\epsilon) \gg 1$

$$\frac{1}{h} m_3\langle|w_R - f - hQ[f]|\rangle = m_3\langle|Q[f] - Q[f_R]|\rangle \le A_1\, m_3\langle|f - f_R|\rangle^{\frac{1}{7}} + A_2\, m_3\langle|f - f_R|\rangle \le \epsilon$$

Hence, w_R is also in $\quad B(f + hQ[f], h\epsilon), \ 0 < h < h_*$ for $R = max\{R(f), R(\epsilon)\}$, then

$$\boxed{B(f + hQ[f], h\epsilon) \cap \mathcal{S}_{\mathfrak{h}_3} \text{ is non-empty for any } 0 < h < h_*.}$$

$\Rightarrow$ The **Sub-tangent condition** holds as

$$h^{-1}\mathrm{dist}(f + hQ[f], \mathcal{S}_{\mathfrak{h}_3}) \le \epsilon, \quad \forall\, 0 < h < h_*, \text{ holds when taking } \quad h_* := \min\{1, h_1(f_{\mathfrak{h}_3, R(f,\epsilon)})\}$$

Uquiness from the One-sided Lipschitz condition:

We need to show $\quad \left[Q[f] - Q[g], f - g\right] \leq C\|f - g\|, \ \forall f, g \in \mathcal{S}$

$$\text{where } \left[\varphi, \phi\right] := \lim_{h \to 0^-} h^{-1}\big(\|\phi + h\varphi\| - \|\phi\|\big).$$

Using dominate convergence theorem one can show that

$$\left[\varphi, \phi\right] \leq \int_{\mathbb{R}^3} \mathrm{d}p \, \varphi(p) \mathrm{sign}(\phi) |p| \, .$$

$\Rightarrow$ the one-side Lipschitz condition by the following lemma showing a **Lipschitz condition for quantum-Boltzmann operator**.

$\Rightarrow$ **Uniqueness** in the same spirit of the original Di Blassio'82 uniqueness proof for initial value problem to the homogeneous Boltzmann equation for hard spheres, using data with enough initial moments.

Lemma: Assume $f, g \in \mathcal{S}$. Then, there exists constant $C := C(\mathfrak{h}_3, \mathfrak{h}_{10}) > 0$ such that

$$\int_{\mathbb{R}^3} dp \, \big(Q[f](p) - Q[g](p)\big) sign(f - g) \big(|p|^1 + |p|^2\big) \leq C \, m_3 \langle |f - g| \rangle \, .$$

Mittag-Leffler moments for solution to the Cauchy problem for the QBE-LT -Partiii

(R.Alonso, IMG, MB.Tran on ArXiv'16)

Propagation and Generation of Mittag-Leffler moments of order $a \in [1,\infty)$ and rate $\alpha > 0$.

In terms infinite sums Taskovic, Alonso, IMG, Pavlovoc'15 it is equivalent to control the integral

$$\int_{\mathbb{R}^3} \mathrm{d}p\, f(t,p)\mathcal{E}_a(\alpha^a|p|) = \sum_{k=1}^{\infty} \frac{\mathcal{M}_k(t)\alpha^{ak}}{\Gamma(ak+1)}\,,$$

where

$$\mathcal{E}_a(x) := \sum_{k=1}^{\infty} \frac{x^k}{\Gamma(ak+1)} \approx e^{x^{1/a}} - 1\,, \qquad x \gg 1\,.$$

(We exclude the term $k=0$ to account for the fact that QBE-LT does not conserves mass)

Theorem **Propagation of Mittag-Leffler tails** Let f be a solution of QBE-LT in $\mathcal{S}$ associated to the initial condition $f_0 \geq 0$, $a \in [1,\infty)$, and suppose that there exists positive α_0 such that

$$\int_{\mathbb{R}^3} dp\, f_0(p)\, \mathcal{E}_a(\alpha_0^a|p|) \leq 1\,.$$

Then, there exists positive constant $\alpha := \alpha(\mathcal{M}_1(0), \alpha_0, a)$ such that

$$\int_{\mathbb{R}^3} dp\, f(t,p)\, \mathcal{E}_a(\alpha^a|p|) \leq 2\,, \qquad \forall\, t \geq 0\,. \tag{0.1}$$

Mittag-Leffler moments for solution to the Cauchy problem for the QBE-LT (R.Alonso, IMG, MB.Tran on ArXiv'16)

- **Lemma** (**Combinatorial sums**) Let $k \geq 3$, then for any $a \in [1, \infty)$,

$$\sum_{i=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} B\big(ai+1, a(k-i)+1\big) \leq C_a (ak)^{-1-a}, \quad \text{for some constant } C_a$$

- **Control of renormalized moments partial sums**

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{k=k_0}^{n} \frac{\mathcal{M}_k\, \alpha^k}{\Gamma(ak+1)} \leq C_1 \sum_{k=k_0}^{n} \sum_{i=1}^{\left[\frac{k+1}{2}\right]} \binom{k}{i} \Big(\mathcal{M}_{i+2} \mathcal{M}_{1+(k-i)} + \mathcal{M}_{i+1} \mathcal{M}_{2+(k-i)} \Big) \frac{\alpha^k}{\Gamma(ak+1)} - C_2 \sum_{k=k_0}^{n} \frac{\mathcal{M}_{k+6}\, \alpha^k}{\Gamma(ak+1)} \,.$$

- **Uniform in time Control of the Mittag-Leffler moment**

$$\int_{\mathbb{R}^3} \mathrm{d}p\, f(t,p) \mathcal{E}_a(\alpha^a |p|) = \lim_{n \to \infty} \mathcal{E}_a^n(\alpha, t) \leq 2 \,, \quad t > 0$$

Mittag-Leffler moments for solution to the Cauchy problem for the QBE-LT

(R.Alonso, IMG, MB.Tran on ArXiv'16)

- From the moments generation estimates

$$m_k(t) \leq C_k(\mathfrak{h}_3)(1 - e^{-C_k t})^{-\frac{k-3}{6}}, \;\; \forall\, k > 3\,.$$

- **Theorem Generation of exponential tails**

Let f be a positive solution of the QBE-LT in $\mathcal{S}$. Then, there exists constant $\alpha > 0$ depending only on $m_3(0)$ such that

$$\int_{\mathbb{R}^3} dp\, f(t,p)|p| e^{\alpha \min\{1, t^{\frac{1}{6}}\}|p|} \leq \frac{1}{2\alpha}, \quad \forall\, t \geq 0.$$

Looking ahead

- **Coupling to of QBE-LT to NLSE-GP**

- **These techniques apply to models on wave (weak) turbulence theory, stratified fluids, oceanography, atmospheric science and near-resonance** *(some work in progress with M.B.Tran, and Smith)*

- **Space inhomogenous problem**

  periodic spacial domains vs.   all space

  ←→ dissipasion vs scattering ?

  (as in Bardos, I.M.G, Golse & Levermore, '15)

- **Numerical methods**

Thank you very much for your attention!

For details: www.ma.utexas.edu/users/gamba/research and references therein